INCENTIVE COMPATIBLE INFORMATION DISCLOSURE

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Incentive Compatible Information Disclosure*

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Abstract

This paper studies the optimal disclosure of information about an agent's quality when it is a combination of a component privately observed by the agent and another latent component. Upon soliciting a report from the agent about his private observation, a principal performs a test which reveals the latent component. The principal then discloses information to the market/public which rewards the agent with compensation equal to the agent's expected quality. We study incentive compatible disclosure rules that minimize the mismatch between the agent's true and expected qualities while inducing truth-telling from the agent. The optimal rule entails full disclosure when the agent's quality is a supermodular function of the two components, but entails partial pooling when it is submodular. We express the optimization problem as a linear transformation of the mean dual-belief, which describes the joint distribution of prior and mean posterior beliefs under disclosure, and obtain the optimal disclosure rule as a corner solution to this linear problem. We identify the number of messages required under the optimal rule and relate it to the agent's incentive compatibility conditions.

Key words: quality, mechanism, revelation, pooling, separating.
JEL Codes: C72, D47, D82.

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1 Introduction

The quality of a product, productivity of a worker, and profitability of a startup are all determined by a combination of multiple attributes some of which are privately observed by the producer, worker or entrepreneur themselves, while other are latent characteristics that can be discovered through the involvement of a third party. For example, a worker's quality or a student's employability depends on a blend of hard skills such as technical expertise and knowledge, and soft skills such as interpersonal skills. While the worker may privately recognize his hard skills, his soft skills may be discovered only in practical work environments. Likewise, a product's quality is determined not only by the quality of the material it uses but also by its durability and performance, which can be discovered through extensive testing. In the same vein, the profitability of a startup is determined not only by its technical innovation but also by its marketability, which often requires close evaluation by venture capitalists (Bergemann and Hege, 1998).

This paper develops a model in which an agent's quality is a combination of two components: The first component is observed privately by the agent and the second component is initially unknown to the agent himself. A principal solicits a report from the agent about the realization of the first component, which we call the agent's *perceptible type* (or *p-type* in short), and then performs a test which reveals the second component, which we call the agent's *latent type* (or *l-type* in short). The distribution of the l-type is such that the higher is the agent's p-type, the higher is his l-type in the sense of stochastic dominance. Upon learning the agent's true quality, the principal discloses information to the market/public who compensates the agent with a transfer that equals his expected quality based on the disclosed information. Through information disclosure and adjustment in the test-taking cost for the agent, the principal's mechanism gives the agent an incentive for truth-telling and minimizes the loss which equals the quadratic difference between the agent's true and expected qualities.

Our model is general enough to encompass a few different applications. For example, the principal in our model can be thought of as a public school system that accepts students to different programs and then sends them to the labor market with grade information, or a human resources department within a firm that offers prospective employees different probationary tasks based on their self-reported qualifications before a permanent contract is signed. Our model can also be interpreted as describing product certification where the principal acts as a public agency that conducts product tests for fees based on information provided by the suppliers, and then discloses product information to consumers. Yet another interpretation of our model is provided by stress tests by the banking authority, which evaluate a bank's performance at the time of crisis (Goldstein and Leitner, 2018). In many of these interpretations, it is natural to suppose that there is positive association between the perceptible and latent attributes: The higher is the perceptible attribute, the more likely it is that the latent attribute is also higher: A startup with a more substantial technical innovation is more likely to seize higher market shares and generate higher profits, a bank with solid financial data is better positioned to weather a financial crisis, a car made with high-quality material is more likely to be safer on the road, a student with strong hard skills is more likely to play a more important role in a team, and so on.[1]

In the absence of the agent's incentive problem, full disclosure of the agent's true quality minimizes the principal's loss function since it eliminates any discrepancy between the true and expected qualities. However, full disclosure does not always induce truth-telling from the agent. Our main result highlights the importance of the functional form of the agent's quality in formulating the optimal disclosure mechanism. Specifically, when we say that a disclosure rule is implementable if it induces truth-telling when combined with proper adjustment of the test-taking cost, full disclosure is implementable if the agent's quality is a supermodular function of his p-type and l-type. Intuitively, supermodularity implies that the higher is the agent's p-type, the larger the marginal impact of the l-type on his quality. For example, the function is supermodular if the quality equals the product or sum of the agent's p-type and l-type. Conversely, the quality function is submodular if the higher is the agent's p-type, the smaller the marginal impact of the l-type on his quality. With a strictly submodular quality function, it is not possible to adjust the test-taking cost to make full disclosure incentive compatible.

To understand the intuition behind the need for pooling, consider the simplest 2×2 environment in which the agent's p-type and the l-type are both binary. We call a pair of the agent's p-type and his l-type a *profile* so that there are four profiles in this case. By assumption, the distribution of the l-type of the high p-type agent stochastically dominates that of the low p-type agent. Since the high p-type agent is more confident in realizing the high l-type than the low p-type agent, the expected quality following the realization of the high l-type is more important for the high p-type than for the low p-type. Conversely, the expected quality following the low l-type is more important for the low p-type than for the high p-type. This suggests that a disclosure rule is implementable (i.e., can be made incentive compatible with adjustment in the test-taking cost) if it creates a larger difference in the expected qualities following the realizations of low and high l-types when the reported p-type is high than when the reported p-type is low. If the agent's quality function is supermodular, this is achieved by perfectly revealing each profile. On the other hand, when it is submodular, full

[1] Several studies emphasize the complementarity between technical knowledge and soft skills (Weinberger, 2014; Balcar, 2016; Piopiunik et al., 2020). For instance, Weidmann and Deming (2021) demonstrates that individuals with strong social or "non-cognitive" skills enhance the productivity of their teams.

disclosure fails to achieve this, and there should be some message that does not fully reveal them. One way to create such a message is to pool two l-types when the reported p-type is low while separating them when the reported p-type is high. Appropriate adjustment in the test-taking cost then induces the agent to report his p-type truthfully. Another way to create an imperfect message is to pool the two p-types when the l-type is low while separating them when the l-type is high. This also makes the expected quality differential higher when the agent reports the high p-type than when he reports the low p-type. In other words, if we define the *ex post compensation function* to be the agent's expected quality as a function of the profile, then the disclosure rule can be made incentive compatible if and only if it renders the ex post compensation function supermodular. Although the argument so far assumes pure disclosure rules that combine full disclosure and complete pooling, the optimal disclosure rule typically entails partial pooling in order to reduce the mismatch loss from pooling. This implies that each profile is perfectly revealed with positive probability, and one pooling message is sent with positive probability based on particular profile realizations. In other words, in the 2×2 environment under consideration, the optimal rule typically sends four perfectly revealing messages and one pooling message. The above discussion also suggests that the pooling message has binary support and is sent only after the realization of certain profiles.

The key step in the analysis is the introduction of the *mean dual-belief,* a joint distribution over pairs of profiles where the first profile is distributed according to the prior and the second is distributed according to the mean posterior beliefs conditional on the first profile. We show that both the principal's objective function and the inequalities expressing implementability are linear transformations of the mean dual-belief. The transformed problem hence has a corner solution. We further show that the number of strictly positive entries in the corner solution equals the number of inequalities in the implementability conditions. Finally, we show that this optimal solution corresponds to the optimal disclosure rule provided that the solution is replicated by the probabilities of pooling messages. In general, this last step requires that the degree of submodularity of the quality function be not too large.

In the 2×2 environment described above, supermodularity of the ex post compensation function is expressed by a single condition: The difference in the expected qualities of the agent with the low and high l-types is higher when his p-type is high than when it is low. The number of pooling messages under the optimal disclosure rule hence equals the number of the conditions required for the supermodularity of the ex post compensation function. In the $K \times 2$ environment where there exist $K \geq 3$ p-types and two l-types, a disclosure rule can be made incentive compatible with the adjustment in the test-taking cost if and only if it leads to a supermodular ex post compensation function as in the 2×2 environment. Since the l-type is binary, the supermodularity of the ex post compensation function reduces to the $K - 1$ local conditions. When the degree of submodularity of the quality function is mild, we show that $K - 1$ gives an upper bound on the number of pooling messages in the optimal disclosure rule. In the general $K \times L$ environment where there exist K p-types and L l-types, the feasibility conditions amount to *cyclical monotonicity* of the *interim compensation function,* which equals the agent's expected quality as a function of his true and reported p-types.[2] Again under a mild degree of submodularity of the quality function, we show that the number of conditions required for cyclical monotonicity gives an upper bound on the number of pooling messages under the optimal disclosure rule.

The paper is organized as follows: Section 2 discusses the related literature. We formulate our model in Section 3 and discuss the implementability of a disclosure rule in Section 4. Section 5 establishes the optimality of full disclosure when the quality function is supermodular and presents the linear transformation of the objective function and implementability conditions using the mean dual-belief. The optimal disclosure rule under a submodular quality function is studied in Section 6 for binary l-types, and in Section 7 for the case where the l-type can take three or more values. We conclude with a discussion in Section 8. Appendix A presents formal derivation of the mean dual-belief and Appendix B collects the proofs of lemmas and propositions. Online appendices present variations of the baseline model: Appendix C presents a model in which the agent can make an ex ante action choice that stochastically enhances his quality and characterizes the optimal disclosure rule that induces the agent to take such an action. Appendix D presents a model in which the principal's objective is to maximize the probability that the agent receives a compensation.

2 Related Literature

The present paper is related to a few strands of the literature. First, it is related to the literature on career concerns (see, e.g., Holmström, 1999; Dewatripont et al., 1999*a*,*b*; Bonatti and Hörner, 2017) which is characterized by the presence of a worker's latent attribute and public information. As in models of career concerns, the agent's incentive in the present model is guided by the market expectation of his quality. However, our approach differs significantly in two ways: instead of focusing on moral hazard, we introduce adverse selection by assuming that the agent privately observes part of his quality. Another important difference is that

[2]The concept originates with Rochet (1987).

we assume that the principal filters information released to the market.[3]

Second, our model is related to the extensive literature on certification design.[4] In canonical models of certification design, a supplier privately informed about the quality of his good chooses whether or not to have it certified, and the certifier optimizes over certification schemes by controlling information disclosed to the public. The key difference in our model hence is the presence of the test stage, which would correspond to product testing in the certification framework.[5,6] One salient conclusion in the certification design literature is that information revealed by the optimal disclosure rule is coarse. For example, it is often found that the binary "pass-fail" scheme is optimal. Harbaugh and Rasmusen (2018) show that the certifier finds it optimal to disclose coarse information when the supplier's reporting incentive is taken into account and the certifier minimizes quadratic loss function. In contrast, the optimal disclosure rule identified in this paper is not coarse in the sense that it involves perfectly revealing messages and also pools at most two profiles under one message even when pooling is required.[7]

The present model can be interpreted as combining the models of career concerns and certification: It introduces the discovery of a latent attribute into the models of certification, and introduces reporting of a privately perceptible attribute into the models of career concerns.

Third, our model is related to the literature on school design which discusses the grading system of a school as a way to disclose information to a potential employer of its students. Among them, Bizzotto and Vigier (2021) present a moral hazard model that combines types and performance. In their setup, a planner observes a students type and designs a grading system that maximizes total educational output, which equals the number of students achieving competency. While Bizzotto and Vigier (2021) assume that competency is a function of both the students type and effort, we assume that a student's marketable competency depends on two inherent characteristics and that schooling offers a process through which his

[3] Rodina (2020) studies a career concerns model in which the principal engages in information disclosure with the objective of maximizing an agent's effort.

[4] See Dranove and Jin (2010) for a comprehensive survey.

[5] There is also difference in the certifier's objective function: In many models, certifiers either maximize their own profit (see, e.g., Lizzeri, 1999), or the senders' benefit as in the case of Ostrovsky and Schwarz (2010) where colleges maximize the students' job prospects.

[6] Some recent literature on certification introduces testing by a certifier. See for example Bizzotto et al. (2020), where the principal decides whether or not to conduct a test to find out the latent type of the agent after observing the experiment designed by the agent himself and its outcome.

[7] The optimality of coarse information in the certification literature is based on different modelling assumptions that are not readily comparable to those of the present model. For example, Harbaugh and Rasmusen (2018) assume that the certifier sets a single fee for the test and perfectly observes the agent's private type when he receives certification.

latent characteristic is discovered. Our model thereby contributes to this literature by analyzing the interaction of multiple competency attributes in the design of an optimal grading system.

Finally, with the principal being the sender of information about the agent's quality, our model belongs to the extensive literature on information design as pioneered by Kamenica and Gentzkow (2011).[8] In contrast with the standard assumption in the literature that the sender has free access to information that he discloses, we suppose that the principal as the sender must collect part of his information from the agent through the provision of proper incentives.[9] In the standard information design problems, the distribution of posterior beliefs is constrained only by the Bayes consistency (plausibility) condition which asserts that the expected value of the posterior beliefs equals the prior belief, enabling the concavification argument.[10] Introduction of the agent's incentive constraints implies major technical differences from the standard environment in that they place additional restrictions on the distribution of posterior beliefs. This makes it difficult to work directly with the distribution of posterior beliefs and prompts an alternative approach. The use of the mean dual-belief, which has not been discussed in the literature to the best of our knowledge, is one such alternative.[11]

3 Model

There exist an agent whose quality θ consists of two components s and ω. The first component s is observed privately by the agent, and is referred to as the agent's *p-type* where p stands for "perceptible." The p-type s is distributed over a finite set $S \equiv \{s_1, \dots, s_K\}$ where $s_1 < \cdots < s_K$ and $K \geq 2$. On the other hand, the second component ω of the quality θ is initially unknown to the agent himself, and is referred to as the agent's *l-type* where l stands for "latent." The l-type ω is distributed over a finite set $\Omega \equiv \{\omega_1, \dots, \omega_L\}$. The agent's quality θ is a non-negative increasing function of s and ω:

$$\theta = \theta(s, \omega).$$

[8] See Kamenica (2019) for an early survey.

[9] Costly information acquisition by the sender is studied in Gentzkow and Kamenica (2014). The sender in their model however does not face incentive issues in the process.

[10] The concavification principle asserts that the highest payoff that the principal can achieve at any prior belief μ^0 equals the weighted average of the payoffs he can achieve at different priors provided that μ^0 equals the same weighted average of those different priors. See Section 8.

[11] Kolotilin et al. (2022) consider the joint distribution of the true state and the receiver's action, which in the present model equals the joint distribution of the profile and the posterior mean. On the other hand, the mean dual-belief is a joint distribution over pairs of profiles. See Appendix A for discussion on the relationship.

The pair (s, ω) is referred to as a *profile* and denoted by v. For any profile $v = (s, \omega)$, write $\theta_v = \theta_{s\omega} = \theta(s, \omega)$. Every profile $v = (s, \omega)$ occurs with strictly positive probability

$$p_v \equiv \Pr(v) > 0 \quad \text{for every } v = (s, \omega) \in V \equiv S \times \Omega,$$

and the conditional distribution $g_s(\omega) \equiv \Pr(\omega \mid s)$ is ordered by stochastic dominance: For any s, $t \in S$ such that $s < t$,

$$\Pr(\omega \leq \omega_\ell \mid s) > \Pr(\omega \leq \omega_\ell \mid t) \quad \text{for } \ell = 1, \ldots, L-1. \tag{1}$$

In other words, a higher p-type s is more likely associated with a higher l-type ω.[12]

The principal elicits from the agent his p-type s and then subjects him to a test which reveals to the principal the agent's l-type ω. The agent incurs test-taking cost y which is a function of his reported p-type. Let $y : S \to \boldsymbol{R}$ be a *cost assignment rule* which specifies the test-taking cost for each reported p-type. At the completion of the test, the principal chooses a message as a function of the profile $v = (s, \omega)$ and sends it to the market/public, which doesn't directly observe either the agent's reported p-type or the test outcome. A *disclosure rule* (Z, f) specifies the functional relationship between the profile $v = (s, \omega)$ and the message: Z is the set of possible messages and $f : V \to \Delta Z$ maps each profile $v = (s, \omega)$ to a probability distribution over Z. Specifically, $f(z \mid v) \in [0, 1]$ is the probability that message z is sent when the profile $v \in V$ is realized. Given a disclosure rule (Z, f), define $\zeta_z \in \Delta V$ to be the posterior belief over profiles conditional on message $z \in Z$, and μ_z to be the expected quality of the agent according to ζ_z:

$$\mu_z = \sum_{v \in V} \theta_v \, \zeta_z(v).$$

The agent then receives from the market/public compensation W equal to his expected quality:

$$W = \mu_z.$$

The agent's utility equals the compensation minus the test-taking cost: $W - y$. A *disclosure mechanism* $\Gamma = (y, Z, f)$ is a pair of the cost assignment rule y and the disclosure rule (Z, f). The timing of events is summarized as follows:

1. The principal chooses and publicly announces the mechanism $\Gamma = (y, Z, f)$.

[12] See the discussion in the Introduction for motivation behind this assumption.

2. The agent observes his p-type s and reports it to the principal.
3. The agent takes the test by incurring the cost $y(s)$ and the principal observes the l-type ω.
4. The principal sends a message z to the market/public according to the disclosure rule (Z, f).
5. The agent receives compensation W equal to his expected quality μ_z.

We now describe the conditions that incentivize the agent to report his p-type truthfully to the principal. The *ex post compensation function* $\phi : V \to \boldsymbol{R}_+$ is defined by

$$\phi(v) = \sum_{z \in Z} \mu_z \, f(z \mid v) \quad \text{for } v \in V.$$

When $v = (s, \omega)$, $\phi(v)$ is the expected quality of the agent (and hence his expected compensation) when he reports p-type s and realizes the l-type ω. Define also the *interim compensation function* $H : S^2 \to \boldsymbol{R}_+$ by

$$H(s, t) = \sum_{\omega \in \Omega} g_s(\omega) \, \phi(t, \omega) \quad \text{for } s,\, t \in S.$$

$H(s, t)$ is the expected quality of the agent before the realization of the l-type ω but after the agent learns his p-type s and reports t to the principal.

The mechanism $\Gamma = (y, Z, f)$ is *incentive compatible* (IC) if the agent has incentive to report his p-type truthfully:[13]

$$H(s, s) - y(s) \geq H(s, t) - y(t) \quad \text{for any } s,\, t \in S. \tag{2}$$

The principal chooses a mechanism to best inform the market/public about the agent's quality. Specifically, the principal aims to minimize the quadratic difference between the agent's quality as revealed from $v = (s, \omega)$, and the market expectation

[13] When the agent's outside option is normalized to zero, Γ is *individually rational* (IR) if $H(s, s) - y(s) \geq 0$ for any $s \in S$. Although inclusion of IR is possible, we ignore it in our analysis to focus on IC.

of his quality formed from the principal's announcement:[14,15]

$$\mathcal{L}(\Gamma) = E_{v,z}\big[(\theta_v - \mu_z)^2\big] = \sum_v \sum_z p_v\, f(z \mid v)\, (\theta_v - \mu_z)^2. \tag{3}$$

The mechanism $\Gamma^* = (y, Z, f)$ is *optimal* if it minimizes $\mathcal{L}(\Gamma)$ in the class of incentive compatible mechanisms:

$$\Gamma^* \in \operatorname{argmin}\{\mathcal{L}(\Gamma) : \Gamma \text{ satisfies (IC)}\}.$$

Since the cost of taking the test does not enter the principal's objective function, it is used solely for the purpose of controlling the agent's incentive in the reporting stage. Our primary focus is hence on the disclosure rule which constitutes an incentive compatible mechanism when coupled with *some* cost assignment rule. Specifically, a disclosure rule (Z, f) is *implementable* if there exists a cost assignment rule y such that the mechanism $\Gamma = (y, Z, f)$ is incentive compatible (IC).

4 Implementable disclosure rules

We begin with the characterization of implementable disclosure rules in terms of the interim compensation function H. The function $H : S^2 \to \boldsymbol{R}$ is *cyclically monotone* if, for any $n = 2, \ldots, K$ and any $k_1, \ldots, k_n \in \{1, \ldots, K\}$ which are all distinct, $k_1 = \min_i k_i$, and $k_0 = k_n$,

$$\sum_{i=1}^{n} \{H(s_{k_i}, s_{k_i}) - H(s_{k_i}, s_{k_{i-1}})\} \geq 0. \tag{4}$$

Cyclical monotonicity, first proposed by Rochet (1987), is illustrated in Figure 1, where $a = H(s_1, s_2) - H(s_1, s_1)$, $b = H(s_2, s_4) - H(s_2, s_2)$, $c = H(s_3, s_3) - H(s_3, s_1)$, and $d = H(s_4, s_4) - H(s_4, s_3)$. The inequality (4) for the sequence $(k_1, k_2, k_3, k_4) = (1, 3, 4, 2)$ requires that $c + d \geq a + b$. In general, the graphical interpretation of (4) is that when we draw a series of (non-overlapping) right triangles with their

[14] This loss function encompasses a preference for conveying accurate information, a reputational incentive if the principal relies on an external certifier agency, or a concern for the welfare of the firm if the principal relies on a specific department within the firm.

[15] Note that (3) can equivalently be written as $\mathcal{L}(\Gamma) = E_z\big[\mathrm{Var}(\theta_v|z)\big]$, where $\mathrm{Var}(\theta_v|z)$ is the posterior variance of θ_v given $z \in Z$. Hence, the principal's objective is alternatively minimization of the expected value of the posterior variance. Since $E_{z,v}[(\theta_v - \mu_z)^2] = E_v[\theta_v^2] - E_z[\mu_z^2]$, the objective is also equivalent to the maximization of $E_z\big[\mu_z^2\big]$. The objective function is hence a function of the posterior mean of the state, a common assumption in the literature (e.g., Kolotilin, 2018; Dworczak and Martini, 2019; Arieli et al., 2023). However, the mean-preserving spread (contraction) technique frequently used in this environment cannot be applied here because of the implementability requirements.

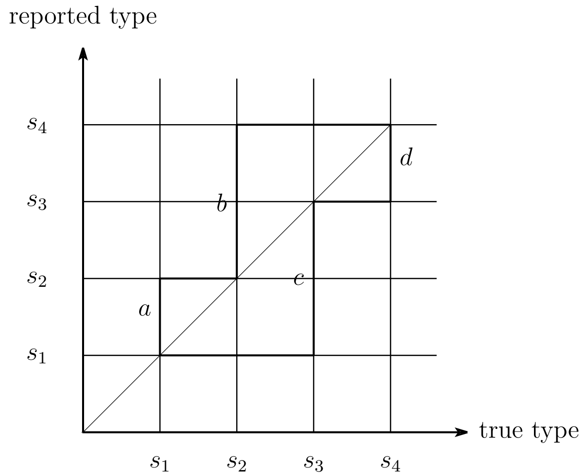

Figure 1: Cyclical monotonicity of H requires $a+b \leq c+d$ when $(k_1, k_2, k_3, k_4) = (1, 3, 4, 2)$ in (4)

hypotenuses on the diagonal both above and below it, the sum of the changes in the value of H along the vertical line segments of those triangles above the diagonal ($a+b$ in the example) is no larger than the sum of the corresponding changes below the diagonal ($c+d$ in the example). Different sequences $(k_1, \ldots, k_n)$ correspond to different collections of such triangles. Since k_1 is chosen to be the smallest among $k_1, \ldots, k_n$, there exist $(n-1)!$ such sequences for a fixed n. The total number of inequalities in (4) for $n = 2, \ldots, K$ hence equals

$$N = \sum_{n=2}^{K} \binom{K}{n} (n-1)!. \tag{5}$$

Importantly, cyclical monotonicity is weaker than supermodularity, which requires that the change in the value of H along each vertical line segment be no larger than the corresponding change along the vertical line segment to the right (Figure 1). The following lemma is a formal statement of this observation.

Lemma 1 *If H is supermodular, then it is cyclically monotone.*

To understand the relationship between cyclical monotonicity of H and implementability of a mechanism, suppose first that the agent has two p-types: $S = \{s_1, s_2\}$. In this case, the unique relevant sequence is $(k_1, k_2) = (1, 2)$, and (4) is written as

$$H(s_1, s_2) - H(s_1, s_1) \leq H(s_2, s_2) - H(s_2, s_1).$$

We can then choose the cost assignment rule y to satisfy

$$H(s_1, s_2) - H(s_1, s_1) \leq y(s_2) - y(s_1) \leq H(s_2, s_2) - H(s_2, s_1).$$

It can be readily verified that these inequalities correspond to the (IC) conditions (2) for s_1 and s_2. Suppose next that the agent has three p-types $S = \{s_1, s_2, s_3\}$. Under Γ, s_1 should have no incentive to misrepresent himself as s_2, s_2 as s_3, and s_3 as s_1. These conditions can be respectively written as:

$$\begin{aligned} y(s_2) - y(s_1) &\geq H(s_1, s_2) - H(s_1, s_1), \\ y(s_3) - y(s_2) &\geq H(s_2, s_3) - H(s_2, s_2), \\ H(s_3, s_3) - H(s_3, s_1) &\geq y(s_3) - y(s_1). \end{aligned}$$

Adding these inequalities side by side, we obtain

$$H(s_3, s_3) - H(s_3, s_1) \geq H(s_1, s_2) - H(s_1, s_1) + H(s_2, s_3) - H(s_2, s_2),$$

which is equivalent to (4) for the sequence $(k_1, k_2, k_3) = (1, 3, 2)$. Different sequences appearing in (4) likewise correspond to the feasibility of different combinations of incentive conditions. The following proposition, which reproduces Rochet (1987, Theorem 1, p192), shows that cyclical monotonicity is not only necessary but also sufficient for the existence of a cost assignment rule y that makes Γ incentive compatible.

Proposition 2 *The disclosure rule (Z, f) is implementable if and only if H is cyclically monotone.*[16]

The following lemma shows that the cyclical monotonicity of the interim compensation function H reduces to the supermodularity of the ex post compensation function ϕ when the l-type is binary ($\Omega = \{\omega_1, \omega_2\}$).

Lemma 3 *When the l-type is binary $\Omega = \{\omega_1, \omega_2\}$, the disclosure rule (Z, f) is implementable if and only if ϕ is supermodular:*

$$\phi(t, \omega_2) + \phi(s, \omega_1) \geq \phi(t, \omega_1) + \phi(s, \omega_2) \quad \text{if } s < t. \tag{6}$$

Furthermore, ϕ is supermodular if and only if

$$\phi(s_{k+1}, \omega_2) + \phi(s_k, \omega_1) \geq \phi(s_{k+1}, \omega_1) + \phi(s_k, \omega_2) \quad \text{for } k = 1, \ldots, K-1. \tag{7}$$

[16]See Rochet (1987) for the proof.

The discussion so far applies generally to any expected payoff functions H and ϕ of the agent and does not depend on the particular interpretation that they represent expected compensation. We below present a few examples that illustrate the implementability of some disclosure rules in the simple 2×2 environment: $S = \{s_1, s_2\}$ and $\Omega = \{\omega_1, \omega_2\}$. Denote the four profiles by

$$v_1 = (s_1, \omega_1), \quad v_2 = (s_1, \omega_2), \quad v_3 = (s_2, \omega_1), \quad \text{and} \quad v_4 = (s_2, \omega_2), \tag{8}$$

and let

$$p_m = p(v_m), \quad \theta_m = \theta(v_m), \quad \phi_m = \phi(v_m) \quad \text{for } m = 1, \ldots, 4. \tag{9}$$

Example 1 (Full disclosure) Let $Z = V$ and

$$f(v_i \mid v_i) = 1 \quad \text{for } i = 1, \ldots, 4.$$

This is the "full" disclosure rule that perfectly reveals every profile (Figure 2). Since $\phi(v_i) = E_v[\theta_v \mid v_i] = \theta_i$ under this rule, it is implementable if and only if θ is supermodular by Lemma 3.

Since the full disclosure rule eliminates any loss arising from the difference between the true and expected qualities, it is clearly optimal for the principal if it is implementable. In the following examples, then, suppose that θ is not supermodular:

$$\Delta \equiv \theta_2 + \theta_3 - \theta_1 - \theta_4 > 0. \tag{10}$$

Example 2 (Pass-Fail 1) Let $Z = \{z_1, z_2\}$, and

$$f(z_1 \mid v) = \begin{cases} 1 & \text{if } v = v_1, \\ 0 & \text{otherwise,} \end{cases} \qquad f(z_2 \mid v) = \begin{cases} 0 & \text{if } v = v_1, \\ 1 & \text{otherwise.} \end{cases}$$

This is the rule where the agent obtains the "Fail" grade z_1 when the profile $v_1 = (s_1, \omega_1)$ realizes and the "Pass" grade z_2 otherwise (Figure 2). In this case, the ex post compensation function ϕ is given by

$$\phi_m = \begin{cases} \theta_1 & \text{if } m = 1, \\ \mu_{-1} & \text{otherwise,} \end{cases}$$

where $\mu_{-1} = E_\theta[\theta \mid v \neq v_1]$. Since $\mu_{-1} > \theta_1$, ϕ is not supermodular:

$$\phi_4 - \phi_3 - \phi_2 + \phi_1 = \theta_1 - \mu_{-1} < 0.$$

It follows that this disclosure rule is *not* implementable.

Example 3 (Pass-Fail 2) Let $Z = \{z_1, z_2\}$, and

$$f(z_1 \mid v) = \begin{cases} 0 & \text{if } v = v_4, \\ 1 & \text{otherwise,} \end{cases} \quad f(z_2 \mid v) = \begin{cases} 1 & \text{if } v = v_4, \\ 0 & \text{otherwise.} \end{cases}$$

This is the rule where the agent obtains the "Pass" grade z_2 when the profile $v_4 = (s_2, \omega_2)$ realizes and the "Fail" grade otherwise (Figure 2). In this case, the ex post compensation function ϕ is given by

$$\phi_m = \begin{cases} \theta_4 & \text{if } m = 4, \\ \mu_{-4} & \text{otherwise,} \end{cases}$$

where $\mu_{-4} = E_v[\theta_v \mid v \neq v_4]$. Since $\mu_{-4} < \theta_4$, ϕ is supermodular:

$$\phi_4 - \phi_3 - \phi_2 + \phi_1 = \theta_{-4} - \mu_1 > 0.$$

This disclosure rule is hence implementable.

The above examples suggest the following intuition: In order to make ϕ supermodular when θ is not, one needs to either "shrink" the distance $\phi_2 - \phi_1$ or $\phi_3 - \phi_1$ by "pushing up" ϕ_1 and at the same time "pulling down" ϕ_2 or ϕ_3 by pooling v_1 with a higher realization v_2 or v_3. The disclosure rule in Example 3 above does this and hence is implementable. On the other hand, the disclosure rule in Example 2 pools the highest profile v_4 with lower realizations. Such a rule is not implementable since it will shrink the distance $\phi_4 - \phi_3$ and $\phi_4 - \phi_2$ and hence lead to an even severer violation of the supermodularity condition.

Example 4 (High-Middle-Low) Let $Z = \{z_1, z_2, z_3\}$, and

$$f(z_1 \mid v) = \begin{cases} 1 & \text{if } v = v_1, \\ 0 & \text{otherwise,} \end{cases} \quad f(z_2 \mid v) = \begin{cases} 1 & \text{if } v = v_2 \text{ or } v_3, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f(z_3 \mid s, \omega) = \begin{cases} 1 & \text{if } v = v_4, \\ 0 & \text{otherwise.} \end{cases}$$

This is the rule where the agent receives the "high" grade z_3 if $v = (s_2, \omega_2)$, the "low" grade z_1 if $v = (s_1, \omega_1)$, and the "medium" grade z_2 otherwise (Figure 2). In this case, the ex post compensation function ϕ is given by

$$\phi_m = \begin{cases} \theta_4 & \text{if } m = 4, \\ \theta_1 & \text{if } m = 1, \\ \mu_{23} & \text{otherwise,} \end{cases}$$

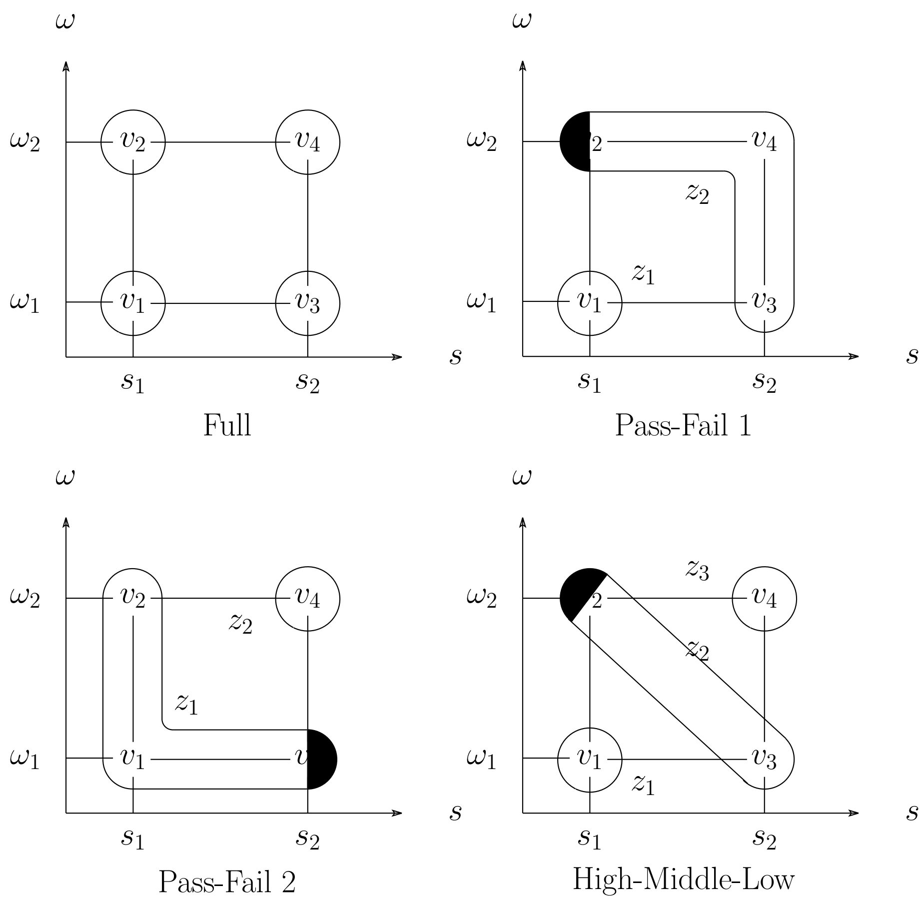

Figure 2: Disclosure rules in the 2×2 environment
Connected profiles are pooled.

Disclosure rule	implementable when $\Delta > 0$?
Full	No
Pass-Fail 1	No
Pass-Fail 2	Yes
High-Middle-Low	Yes iff $\Delta \leq \frac{(\theta_3-\theta_2)(p_2-p_3)}{p_2+p_3}$

where

$$\mu_{23} = E_v[\theta_v \mid v \in \{v_2, v_3\}] = \frac{p_2\theta_2 + p_3\theta_3}{p_2 + p_3}.$$

Since $\theta_1 < \mu_{23} < \theta_4$, this disclosure rule is implementable if and only if

$$\begin{aligned} &\phi_4 - \phi_3 - \phi_2 + \phi_1 = \theta_4 - 2\mu_{23} + \theta_1 \geq 0 \\ \Leftrightarrow \quad &\Delta \leq \frac{(p_2 - p_3)(\theta_3 - \theta_2)}{p_2 + p_3}, \end{aligned} \tag{11}$$

where Δ is as defined in (10). Hence, if

$$(p_2 - p_3)(\theta_3 - \theta_2) > 0, \tag{12}$$

then (11) holds if $\frac{\Delta}{|\theta_3 - \theta_2|} > 0$ is small compared with $|p_2 - p_3|$. We will return to this last observation in Section 6.

5 Full and partial disclosure rules

As noted in Section 4, when there is no incentive issue in the reporting stage, perfectly disclosing information about the realized profile $v = (s, \omega)$ clearly minimizes the principal's loss function $\mathcal{L}$. Specifically, (Z, f) is a *full disclosure* rule if $Z = V$, and

$$f(z \mid v) = \begin{cases} 1 & \text{if } z = v, \\ 0 & \text{otherwise.} \end{cases}$$

The full disclosure rule however may not induce truth-telling from the agent when his p-type s is private. To see when full disclosure induces truth-telling, note that the ex post expected quality ϕ equals the true quality θ under full disclosure, and hence that the interim compensation function H is given by $H(s,t) = \sum_\omega g_s(\omega)\, \theta(t, \omega)$. Even when p-type s is private, hence, full disclosure is implementable if and only if this function is cyclically monotone. The following proposition presents a sufficient condition for this as the first main result on the optimal disclosure rule.

Proposition 4 *Suppose that the quality function θ is supermodular. Then the optimal mechanism Γ entails full disclosure.*

For example, the quality function θ is supermodular if for $\delta \geq 0$,

$$\theta(s, \omega) = s + \omega + \delta\, s\omega, \tag{13}$$

or if $\theta(s,\omega) = \min\{s,\omega\}$. As suggested by these examples, s and ω can be either complements or substitutes when θ is supermodular.[17]

In view of Proposition 4, we assume in what follows that the quality function θ is not supermodular, and more concretely, that it is *strictly submodular*: For any s, $t \in S$ and ω, $\hat{\omega} \in \Omega$ such that $t > s$ and $\hat{\omega} > \omega$,

$$\theta(t,\hat{\omega}) + \theta(s,\omega) < \theta(s,\hat{\omega}) + \theta(t,\omega). \tag{14}$$

For example, θ is strictly submodular if $\delta < 0$ in (13), or if there exists a strictly concave and increasing function $u : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ such that

$$\theta(s,\omega) = u(s+\omega).$$

When considering a class of quality functions in our analysis, we fix the relative ranking of different profiles according to their values. Specifically, we consider a total ordering $\succcurlyeq$ over the set $V = S \times \Omega$ of profiles that are *consistent* with the value of θ in the sense that

$$v \prec \hat{v} \quad \Leftrightarrow \quad \theta(v) < \theta(\hat{v}).$$

Since we assume that θ is increasing, $\succcurlyeq$ satisfies

$$(s,\omega) \leq (\hat{s},\hat{\omega}) \text{ and } (s,\omega) \neq (\hat{s},\hat{\omega}) \quad \Rightarrow \quad (s,\omega) \prec (\hat{s},\hat{\omega}).$$

When $\hat{s} > s$ and $\hat{\omega} > \omega$, any of $(s,\hat{\omega}) \succ (\hat{s},\omega)$, $(\hat{s},\omega) \succ (s,\hat{\omega})$, and $(s,\hat{\omega}) \sim (\hat{s},\omega)$ is possible.

Given a disclosure rule (Z,f), the *mean dual-belief* $\bar{\psi}_f$ is a probability distribution over pairs of profiles $(v,\hat{v}) \in V^2$ given by[18]

$$\bar{\psi}_f(v,\hat{v}) = p_v \operatorname{Pr}_f(\hat{v} \mid v) \quad \text{for every } (v,\hat{v}), \tag{15}$$

where

$$\operatorname{Pr}_f(\hat{v} \mid v) = \sum_{z \in Z} f(z \mid v)\, \zeta_z(\hat{v})$$

is the mean posterior belief weight on $\hat{v}$ conditional on v under (Z,f). In other words, for each v, $\bar{\psi}_f(v,\cdot)/p_v$ is the weighted average of the posterior beliefs ζ_z

[17] For example, s and ω are complements if $\theta = s\omega$ and are substitutes if $\theta = s + \omega$ even though θ is supermodular in both cases. Proposition 4 implies the optimality of full disclosure in a model in which the agent has no latent type since θ is supermodular if $\Omega = \{\omega_1\}$.

[18] Alternatively, $\bar{\psi}_f$ is the mean dual-belief corresponding to the dual-belief distribution τ_f specified in Proposition 10 in Appendix A.

when the weight on each z equals $f(z \mid v)$. As shown in Appendix A, $\bar{\psi}_f(v, \hat{v})$ also equals the mean of the product of posterior beliefs:

$$\bar{\psi}_f(v, \hat{v}) = \sum_{z \in Z} \Big\{ \sum_{v} p_v \, f(z \mid v) \Big\} \, \zeta_z(v) \, \zeta_z(\hat{v}). \tag{16}$$

It follows that $\bar{\psi}_f$ is symmetric ($\bar{\psi}_f(v, \hat{v}) = \bar{\psi}_f(\hat{v}, v)$ for any $(v, \hat{v})$), and the marginal distribution of $\bar{\psi}_f$ equals the prior ($\sum_{\hat{v}} \bar{\psi}_f(v, \hat{v}) = p_v$ for any v).[19]

Lemma 5 *Let $\bar{\psi}_f$ be the mean dual-belief corresponding to a disclosure rule (Z, f). The ex post compensation function ϕ, the interim compensation function H, and the quadratic loss function $\mathcal{L}$ are all linear functions of $\bar{\psi}_f$. Specifically, we have*

$$\phi(v) = \frac{1}{p_v} \sum_{\hat{v}} \bar{\psi}_f(v, \hat{v}) \, \theta_{\hat{v}}, \tag{17}$$

$$H(s, t) = \sum_{\omega} \frac{g_s(\omega)}{p(t, \omega)} \sum_{\hat{v}} \bar{\psi}_f\big((t, \omega), \hat{v}\big) \, \theta_{\hat{v}}, \tag{18}$$

$$\mathcal{L}(\Gamma) = \sum_{v \prec \hat{v}} \bar{\psi}_f(v, \hat{v}) \, (\theta_{\hat{v}} - \theta_v)^2. \tag{19}$$

Lemma 5 presents a key observation that allows us to consider a simpler linear problem that corresponds to the principal's optimization problem.[20] In what follows, we consider disclosure rules with a finite message set Z expressed as follows:

$$\begin{aligned} &Z = V \cup \{z_1, \ldots, z_R\}, \quad V \cap \{z_1, \ldots, z_R\} = \emptyset, \\ &f(v \mid v) + \sum_{r=1}^{R} f(z^r \mid v) = 1 \text{ for any } v \in V. \end{aligned} \tag{20}$$

In other words, when the profile v is realized, one of the $R+1$ messages $v, z_1, \ldots, z_R$ is potentially chosen. Since the message $v \in V$ is sent only after its realization ($f(\hat{v} \mid v) = 0$ if v, $\hat{v} \in V$ and $v \neq \hat{v}$), each $v \in V$ is a perfectly revealing message of its realization.[21] In contrast, each $z^r \in Z$ is a *pooling message* that is sent with positive probability after the realization of multiple profiles.

[19] We can also verify that when (Z, f) is the full disclosure rule, $\bar{\psi}_f$ is diagonal:

$$\bar{\psi}_f(v, \hat{v}) = \begin{cases} p_v & \text{if } v = \hat{v}, \\ 0 & \text{otherwise} \end{cases}$$

and when (Z, f) is the no disclosure rule, $\bar{\psi}_f(v, \hat{v}) = p_v \, p_{\hat{v}}$ for every $(v, \hat{v}) \in V^2$.

[20] See problem (29) in Section 6. As discussed there, we can recover from the solution to this linear problem an optimal disclosure rule through some additional step.

[21] Inclusion of such a message v in Z is without loss of generality since $f(v \mid v) = 0$ is also allowed.

Given any disclosure rule (Z, f) represented as in (20), define for each $v, \hat{v} \in V$ and $r = 1, \ldots, R$,

$$\alpha_v^r = f(z^r \mid v), \quad \text{and} \quad \sigma^r = \sum_{v \in V} p_v \alpha_v^r. \tag{21}$$

As seen, α_v^r is the probability of message z^r conditional on profile v, and σ^r is the marginal probability of z^r. For each message $z^r \in Z$, let $\text{supp}(z^r)$ denote the *support* of z^r:

$$\text{supp}(z^r) = \{v : \alpha_v^r > 0\}.$$

When $v \neq \hat{v}$, $f(v \mid \hat{v}) = 0$ by definition so that the posterior weight on $\hat{v}$ given message $z = v$ equals zero: $\zeta_v(\hat{v}) = 0$. Since $\zeta_{z^r}(v) = \frac{p_v \alpha_v^r}{\sigma^r}$, it follows from (16) that

$$\bar{\psi}_f(v, \hat{v}) = f(v \mid v)\, \zeta_v(v)\zeta_v(\hat{v}) + \sum_r \sigma^r \zeta_{z^r}(v)\zeta_{z^r}(\hat{v}) = p_v p_{\hat{v}} \sum_r \frac{\alpha_v^r \alpha_{\hat{v}}^r}{\sigma^r}.$$

Let now x be defined by

$$x_{v\hat{v}} = |\theta_v - \theta_{\hat{v}}| \frac{\bar{\psi}_f(v, \hat{v})}{p_v\, p_{\hat{v}}} = |\theta_v - \theta_{\hat{v}}| \sum_{r=1}^{R} \frac{\alpha_v^r \alpha_{\hat{v}}^r}{\sigma^r}. \tag{22}$$

By definition, $x_{v\hat{v}} = x_{\hat{v}v}$ for any v, and $\hat{v} \in V$, $x_{vv} = 0$ for every v. ϕ, H and $\mathcal{L}$ are all linear functions of $x = (x_{v\hat{v}})_{v,\hat{v}}$ since they are linear in $\bar{\psi}_f$ by Lemma 5. Our analysis in what follows revolves around variable x instead of the mean dual-belief $\bar{\psi}_f$ itself.[22]

The rest of this section introduces some properties of the quality function θ. Given $\varepsilon > 0$, we say that a submodular quality function θ is *ε-linear* if there exists $h > 0$ such that for any $s < t$ and ω,[23]

$$\left| \frac{\theta(t, \omega) - \theta(s, \omega)}{t - s} - h \right| < \varepsilon. \tag{23}$$

[22]The reason for the use of x is as follows: We will consider optimal disclosure when the degree of submodularity of quality functions θ becomes small, which requires considering a sequence of θ's. The new variable x simplifies this process by allowing us to remove the effects of θ from the left-hand sides of the implementability conditions such as (43) in the proof of Proposition 6. The proof of Lemma 5 also contains the formulae of ϕ, H and $\mathcal{L}$ expressed in terms of x.

[23]When θ is submodular, its degree of submodularity is small if it is also ε-linear since for any $s < t$ and $\omega < \hat{\omega}$,

$$\begin{aligned} \big\{\theta(t, \hat{\omega}) + \theta(s, \omega)\big\} - \big\{\theta(s, \hat{\omega}) + \theta(t, \omega)\big\} &\geq (h - \varepsilon)(t - s) - (h + \varepsilon)(t - s) \\ &= -2\varepsilon(t - s) \geq -2\varepsilon(s_K - s_1). \end{aligned}$$

Given $\eta > 0$, a quality function θ has a *value margin* η if, when θ changes its values from one profile v to another profile $\hat{v}$, it does so at least by margin η: There exists $\eta > 0$ such that for any v, $\hat{v} \in V$,

$$\theta(v) \neq \theta(\hat{v}) \quad \Rightarrow \quad |\theta(v) - \theta(\hat{v})| > \eta.$$

Given $\succcurlyeq$ and $\eta > 0$, let $\Theta_{\succcurlyeq,\eta}$ be the class of quality functions such that

$$\Theta_{\succcurlyeq,\eta} = \{\theta : \theta \text{ is consistent with } \succcurlyeq, \text{ and has a value margin } \eta\}. \tag{24}$$

6 Models with binary *l*-types

A binary specification of the l-type ($L = 2$) is relevant when it can be judged either good or poor because of physical limitation in precise measurement. In this case, a disclosure rule is implementable if and only if the ex post compensation function ϕ is supermodular by Lemma 3. Lemma 3 further shows that submodularity of ϕ reduces to the $(K - 1)$ local conditions (7), implying that implementability of a disclosure rule is expressed by $(K - 1)$ inequalities. We begin with the 2×2 model where $K = L = 2$ (i.e, $S = \{s_1, s_2\}$ and $\Omega = \{\omega_1, \omega_2\}$), and then consider the case where $K \geq 3$.

6.1 2×2 Model

The discussion of implementable disclosure rules in Section 4 (Examples 1-4) already furnishes the key intuitions developed in this section. We use the notation in (8) and (9) in Section 4 while noting that the submodularity of θ in (14) is equivalent to (10). Although none of the disclosure rules in Examples 2-4 involve randomization, the principal may also benefit from randomization if pooling multiple profiles without randomization results in the slackness in the supermodularity of ϕ. This can be seen in the following example.

Example 5 Suppose that the disclosure rule (Z, f) is such that

- $Z = V \cup \{z\}$ for $z \notin V$;
- $f(z \mid v_1) = f(z \mid v_2) = \lambda$ for some $\lambda \in (0, 1)$;
- $f(v_1 \mid v_1) = f(v_2 \mid v_2) = 1 - \lambda$ and $f(v_3 \mid v_3) = f(v_4 \mid v_4) = 1$.

Figure 3 illustrates this disclosure rule, which pools v_1 and v_2 with probability λ, but perfectly reveals them with probability $1 - \lambda$. It also perfectly reveals both v_3

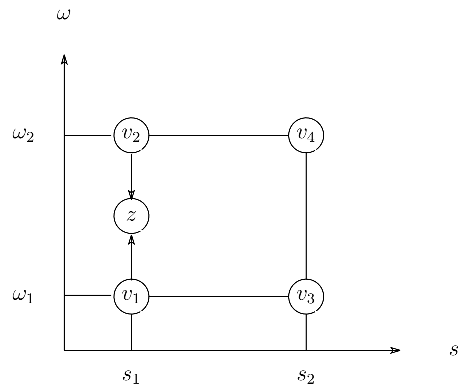

Figure 3: Disclosure rule of Example 5 in the 2×2 environment

Each circle represents a message. z is an pooling message that is sent when either v_1 or v_2 is realized.

and v_4. ϕ is supermodular if and only if

$$\begin{aligned}\phi_1 + \phi_4 - \phi_2 - \phi_3 &= \{\lambda\theta_1 + (1-\lambda)\mu_1\} + \theta_4 - \{\lambda\theta_2 + (1-\lambda)\,\mu_1\} - \theta_3 \\ &= (\theta_4 - \theta_3) - \lambda(\theta_2 - \theta_1) \\ &\geq 0.\end{aligned}$$

When $\lambda = 1$, (Z, f) is the full disclosure rule and not implementable if θ is submodular ($\frac{\theta_4-\theta_3}{\theta_2-\theta_1} < 1$). On the other hand, when $\lambda = 0$, it is implementable but always generates a loss equal to $(\theta_1 - \mu_1)^2$ when the profile is (s_1, ω_1) and $(\theta_2 - \mu_1)^2$ when the profile is (s_1, ω_2). One can minimize the probability of such a loss while maintaining implementability by setting $\lambda = \frac{\theta_4-\theta_3}{\theta_2-\theta_1}$.

Proposition 6 *Suppose that the quality function θ is submodular ($\Delta > 0$) and that (θ, p) satisfies either one of (25), (26), and (27) below:*[24]

$$(p_2 - p_3)(\theta_3 - \theta_2) \leq 0, \tag{25}$$

$$(p_2 - p_3)\left(\frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} - \frac{p_3(p_1 + p_2)}{p_2(p_1 + p_3)}\right) \leq 0, \tag{26}$$

$$\Delta \leq \frac{(p_2 - p_3)(\theta_3 - \theta_2)}{p_2 + p_3}. \tag{27}$$

Then there exists an optimal disclosure rule (Z, f) with exactly one pooling message z:

$$Z = V \cup \{z\} \text{ for } z \notin V.$$

[24]Since (26) implies $(p_2 - p_3)(\theta_3 - \theta_2) > 0$, (25) and (26) are mutually exclusive, and so are (25) and (27). On the other hand, (26) and (27) have an overlap.

The support of the pooling message z is binary and is given by

$$\operatorname{supp}(z) = \begin{cases} \{v_1, v_2\} & \textit{if } \frac{\theta_2-\theta_1}{\theta_3-\theta_1} \le \frac{p_3(p_1+p_2)}{p_2(p_1+p_3)} \textit{ and} \\ & \textit{either } p_2 > p_3 \textit{ or } (p_2-p_3)(\theta_3-\theta_2) \le 0, \\ \{v_1, v_3\} & \textit{if } \frac{\theta_2-\theta_1}{\theta_3-\theta_1} \ge \frac{p_3(p_1+p_2)}{p_2(p_1+p_3)} \textit{ and} \\ & \textit{either } p_2 < p_3 \textit{ or } (p_2-p_3)(\theta_3-\theta_2) \le 0 \\ \{v_2, v_3\} & \textit{if (27) holds but (26) fails.} \end{cases}$$

Proposition 6 shows that the unique pooling message is sent only after the realization of a particular pair of profiles.[25] In line with the intuition provided in Examples 2-5, the pooling shrinks the difference between the expected qualities at v_1 and at v_2 or between those at v_1 and at v_3. The disclosure rule of Example 5 (in Figure 3) is indeed one of the rules described in Proposition 6. Although the highest profile v_4 is never pooled with other profiles, v_2 and v_3 are pooled with each other in some cases.[26]

We also note that the probability $f(z \mid v)$ that the pooling message z is sent increases with Δ: The proof of Proposition 6 shows that the probability of pooling under the optimal disclosure rule can be taken as:

$$\begin{array}{ll} f(z \mid v_1) = f(z \mid v_2) = \frac{\Delta}{\theta_2-\theta_1} & \text{if } \operatorname{supp}(z) = \{v_1, v_2\}, \\ f(z \mid v_1) = f(z \mid v_3) = \frac{\Delta}{\theta_3-\theta_1} & \text{if } \operatorname{supp}(z) = \{v_1, v_3\}, \\ f(z \mid v_2) = f(z \mid v_3) = \frac{p_2+p_3}{p_2-p_3}\frac{\Delta}{\theta_3-\theta_2} & \text{if } \operatorname{supp}(z) = \{v_2, v_3\}. \end{array} \tag{28}$$

The interpretation is that the higher degree of submodularity Δ requires a higher probability of pooling in order to make the ex post compensation function ϕ supermodular.

Graphical illustration of Proposition 6 is possible with the introduction of some structure on p. Let $q = \Pr(s_1) \in (0,1)$ be the probability that the agent has the low p-type, and suppose that

$$\gamma \equiv \Pr(\omega_1 \mid s_1) = \Pr(\omega_2 \mid s_2) > \tfrac{1}{2}.$$

γ is the probability that the l-type is low when the agent has the low p-type or that it is high when the agent has the high p-type. $\gamma > \frac{1}{2}$ ensures first-order stochastic

[25] In other words, the posterior belief given the pooling message is the convex combination of two degenerate posteriors. Kolotilin et al. (2022) call such posteriors *pairwise*.

[26] Since (27) is equivalent to (11), the disclosure rule that pools v_2 and v_3 with positive probability is optimal only if the disclosure rule that pools these profiles with probability one (in Example 5) is implementable.

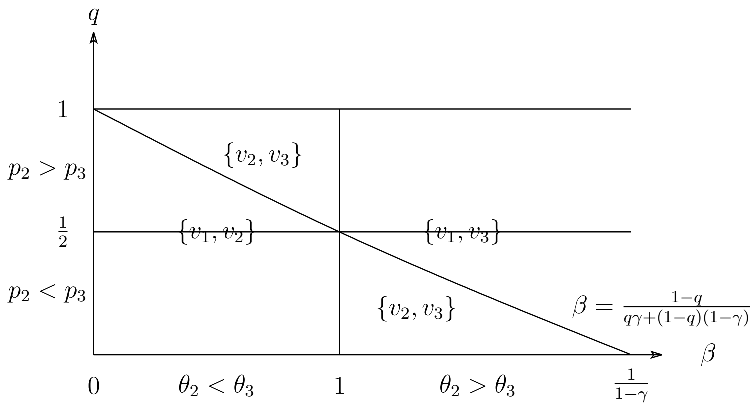

Figure 4: Support of the pooling message z

$q = \Pr(s_1)$, $\beta = \frac{\theta_2-\theta_1}{\theta_3-\theta_1}$, and $\gamma = \Pr(\omega_1 \mid s_1) = \Pr(\omega_2 \mid s_2)$.
$q \lesseqgtr \frac{1}{2} \Leftrightarrow p_2 \lesseqgtr p_3$ and $\beta \lesseqgtr 1 \Leftrightarrow \theta_2 \lesseqgtr \theta_3$.
Pooling with support $\{v_2, v_3\}$ is optimal if and only if (27) holds.

dominance (1).[27] Denote

$$\beta = \frac{\theta_2 - \theta_1}{\theta_3 - \theta_1}.$$

The sufficient conditions of Proposition 6 are then written as:

$$\begin{aligned}
(25) &\quad \Leftrightarrow \quad (1-\beta)(2q-1) \leq 0, \\
(26) &\quad \Leftrightarrow \quad (2q-1)\left(\beta - \frac{1-q}{q\gamma + (1-q)(1-\gamma)}\right) \leq 0, \\
(27) &\quad \Leftrightarrow \quad \Delta \leq (1-\beta)(2q-1).
\end{aligned}$$

Figure 4 describes the optimal disclosure rule for each combination of (β, q). As seen, pooling v_1 and v_2 is optimal when $\beta < 1$ ($\Leftrightarrow \theta_2 < \theta_3$) and $q = \Pr(s_1)$ is not high, and pooling v_1 and v_3 is optimal when $\beta > 1$ ($\Leftrightarrow \theta_2 > \theta_3$) and q is not low. The disclosure rule that pools v_2 and v_3 is feasible only when Δ satisfies (27).

The intuition behind Proposition 6 is as follows: Using x defined in (22), we

[27]The joint distribution p is hence given by

$$p_1 = q\gamma, \quad p_2 = q(1-\gamma), \quad p_3 = (1-q)(1-\gamma), \quad p_4 = (1-q)\gamma.$$

first solve the following problem:[28]

$$\min_x \mathcal{L} \quad \text{subject to:} \begin{cases} \phi_1 + \phi_4 - \phi_2 - \phi_3 \geq 0; & \text{(implementability)} \\ x_{v\hat{v}} = x_{\hat{v}v} \geq 0 \text{ for every } (v, \hat{v}); & \text{(non-negativity)} \\ x_{vv} = 0 \text{ for every } v. & \end{cases} \tag{29}$$

Since the objective function $\mathcal{L}$ and the implementability conditions in terms of the ex post compensation function ϕ are both linear in x, (29) is a linear problem and admits a corner solution x^*. Specifically, corresponding to the single inequality in the implementability condition, there exists a single pair of profiles $(v, \hat{v})$ such that $x^*_{v\hat{v}} > 0$. (Without (IC), $x^* = 0$ is clearly optimal.) By (22), this implies that $\sum_{r=1}^R \alpha^r_v \alpha^r_{\hat{v}} > 0$ for exactly one pair of profiles $(v, \hat{v})$. This pair is shown to be one of $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$. We then recover α^r_v and $\alpha^r_{\hat{v}}$ from $x^*_{v\hat{v}}$ by first setting the number of pooling messages equal to one $R = 1$, and let this pooling message $z = z^1$ be sent only when the realized profile is either v or $\hat{v}$: α^1_v, $\alpha^1_{\hat{v}} > 0$ and $\alpha^1_{v'} = 0$ if $v' \neq v$, $\hat{v}$. When $\{v, \hat{v}\} = \{1, 2\}$ or $\{1, 3\}$, this process always gives rise to legitimate probabilities α^r_v, $\alpha^r_{\hat{v}} \leq 1$. When $\{v, \hat{v}\} = \{2, 3\}$, on the other hand, we need (27) for α_{v_2}, $\alpha_{v_3} \leq 1$ to hold.[29] On the other hand, Proposition 6 holds whenever the quality function θ is mildly submodular so that $\Delta > 0$ is small and satisfies (27). In other words, implementability holds for x^* small ensuring α^r_v, $\alpha^r_{\hat{v}} \leq 1$ even when $\{v, \hat{v}\} = \{2, 3\}$. This observation leads to the following corollary to Proposition 6.

Corollary 7 *Let $\succcurlyeq$ and $\eta > 0$ be given, and suppose that $\theta \in \Theta_{\succcurlyeq,\eta}$ is submodular and ε-linear for ε satisfying $\frac{4\varepsilon}{\eta} \leq \frac{|p_2 - p_3|}{p_2 + p_3}$. Then there exists an optimal disclosure rule as described in Proposition 6.*

In the analysis of a more general environment below, we generalize Corollary 7 by assuming that $\Delta > 0$ is not large while fixing the probability distribution p.

6.2 $K \times 2$ Model

We now suppose that the number $K = |S|$ of the agent's p-types s can be greater than two but continue to assume that the l-type ω is binary.

[28]Note that this problem is only partial since for example it ignores the marginal probability requirement $p_v = \sum_{\hat{v}} \bar{\psi}(v, \hat{v}) = \sum_{\hat{v}} \frac{p_v p_{\hat{v}} x_{v\hat{v}}}{|\theta_v - \theta_{\hat{v}}|}$ ($\Leftrightarrow \sum_{\hat{v}} \frac{p_{\hat{v}} x_{v\hat{v}}}{|\theta_v - \theta_{\hat{v}}|} = 1$) for every v.

[29]This can be seen from the fact that $f(z \mid v_2) = f(z \mid v_3) \leq 1$ in the third line of (28) if and only if (27) holds. When $\{v, \hat{v}\} = \{2, 3\}$ but (27) fails, characterization of the optimal rule is difficult and remains an open question: We conjecture that the optimal disclosure rule involves four fully revealing messages $v_1, \ldots, v_4$ along with a pooling message z with support $\text{supp}(z) = \{v_1, v_2, v_3\}$ as in Example 3.

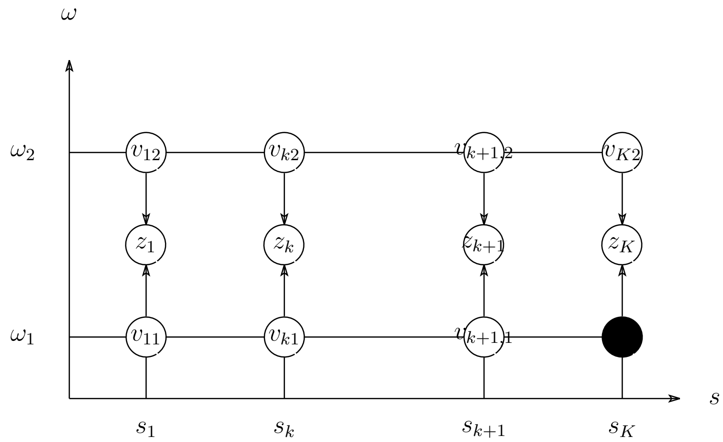

Figure 5: Disclosure rule of Example 6 in the $K \times 2$ environment

Example 6 Consider the following generalization of the disclosure rule discussed in Example 5: (Z, f) is such that for $z_1, \ldots, z_K \notin V$ and $\lambda_1, \ldots, \lambda_K \in [0, 1]$,

- $Z = V \cup \{z_1, \ldots, z_K\}$;
- $f(v_{k\ell} \mid v_{k\ell}) = \lambda_k$ for every k, ℓ;
- $f(z_k \mid v) = \begin{cases} 1 - \lambda_k & \text{if } v = v_{k1} \text{ or } v_{k2}, \\ 0 & \text{otherwise,} \end{cases}$

Figure 5 depicts this disclosure rule, which either perfectly reveals the realized profile or pools the two profiles v_{k1} and v_{k2} with the same p-type s_k if either of them occurs. Define

$$\mu_k = E_v[\theta_v \mid z_k] = \frac{p_{k1}\theta_{k1} + p_{k2}\theta_{k2}}{p_{k1} + p_{k2}}.$$

Then

$$\phi_{k\ell} = \lambda_k \theta_{k\ell} + (1 - \lambda_k)\, \mu_k,$$

so that

$$\phi_{k1} + \phi_{k+1,2} - \phi_{k2} - \phi_{k+1,1} = -\lambda_k(\theta_{k2} - \theta_{k1}) + \lambda_{k+1}(\theta_{k+1,2} - \theta_{k+1,1}).$$

It then follows from (7) that ϕ is supermodular if and only if

$$\frac{\lambda_{k+1}}{\lambda_k} \geq \psi_k \equiv \frac{\theta_{k2} - \theta_{k1}}{\theta_{k+1,2} - \theta_{k+1,1}} \quad \text{for } k = 1, \ldots, K-1. \tag{30}$$

Since θ is submodular, $\psi_k > 1$, suggesting that the probability of perfect revelation of the l-type should increase with k. In particular, we may take $\lambda_K = 1$ so that the l-type is fully disclosed when the agent reports the highest p-type s_K.

Proposition 8 *Let p, $\succcurlyeq$ and $\eta > 0$ be given, and suppose that the quality function $\theta \in \Theta_{\succcurlyeq,\eta}$ is submodular. Then there exists $\varepsilon > 0$ such that if θ is ε-linear, there exists an optimal disclosure rule (Z, f) such that*

- $Z = V \cup \{z_1, \ldots, z_R\}$ *for some* $R \leq K - 1$ *and* $z_1, \ldots, z_R \notin V$.
- $|\operatorname{supp}(z^r)| = 2$ *for every* $r = 1, \ldots, R$.
- $\operatorname{supp}(z^r) \neq \{v_{12}, v_{K2}\}$, $\{v_{K1}, v_{K2}\}$ *for any* $r = 1, \ldots, R$.

Proposition 8 shows that if the quality function θ is mildly submodular, there exists an optimal disclosure rule with at most $(K - 1)$ pooling messages each of which pools no more than two profiles. Furthermore, the pooled pair of profiles is never the combination of the extreme upper-left profile and the extreme upper-right profile (v_{12}, v_{K2}) or the extreme lower-right profile and the extreme upper-right profile (v_{K1}, v_{K2}).

Although Proposition 8 establishes the existence of an optimal disclosure rule with at most $K - 1$ pooling messages, not every optimal disclosure rule needs to have such a property. First, as mentioned in Section 6.1, the argument is based on the existence of a corner solution x^* to the linear problem that has at most $K - 1$ strictly positive coordinates. For a non-generic specification of (p, θ), the linear problem may have multiple (non-corner) solutions which would correspond to more than $K - 1$ pairs of profiles being pooled. Second, the proof of the proposition replicates x^* using $(\alpha_v^r)_{v,r}$ such that for each $r = 1, \ldots, K - 1$, α_v^r, $\alpha_{\hat{v}}^r > 0$ for a single pair $(v, \hat{v})$. There may as well be other ways to replicate x^*. The number of pooling messages can also be strictly less than $K - 1$. To see this point, return to Example 6 and assume $K = 4$. The disclosure rule in this example is in line with the statement of Proposition 8 since it has $K - 1 = 3$ pooling messages each of which has binary support. On the other hand, if a disclosure rule has just one pooling message z_1 which has support $\{v, \hat{v}, \tilde{v}\}$, then this message pools $\binom{3}{2} = 3$ pairs of profiles with each other ($(v, \hat{v})$, $(v, \tilde{v})$ and $(\hat{v}, \tilde{v})$), and hence would imply $x^*_{v\hat{v}} > 0$ for three pairs of profiles $(v, \hat{v})$. Such a rule would also be consistent with the proof of the proposition.

7 General model

We now consider the most general framework where the number of the l-types is greater than two ($L \geq 3$). Unlike when $L = 2$, the ex post compensation function ϕ is not required to be supermodular when $L \geq 3$. Instead, as seen in Proposition 2, the disclosure rule (Z, f) is implementable if and only if the interim compensation function $H : S^2 \to \boldsymbol{R}$ is cyclically monotone. Recall from (5) that

$N \equiv \sum_{n=2}^{K} \binom{K}{n} (n-1)!$ equals the number of inequalities in the definitions of cyclical monotonicity of H.

Example 7 Further generalize the disclosure rule discussed in Example 6 as follows. (Z, f) is such that for $z_1, \ldots, z_K \notin V$ and $\lambda_1, \ldots, \lambda_K \in [0, 1]$,

- $Z = V \cup \{z_1, \ldots, z_K\}$;
- $f(v_{k\ell} \mid v_{k\ell}) = \lambda_k$ for every k, ℓ;
- $f(z_k \mid v) = \begin{cases} 1 - \lambda_k & \text{if } v \in \{v_{k1}, \ldots, v_{kL}\}, \\ 0 & \text{otherwise.} \end{cases}$

In other words, when the profile $v_{k\ell}$ is realized, it is either perfectly revealed or is pooled with *all other* profiles with the same s-coordinates. For any s, $t \in S$, define

$$\nu(s, t) = \sum_{\omega \in \Omega} g_s(\omega)\, \theta_{t\omega}.$$

$\nu(s, t)$ can be interpreted as the interim expected quality under full disclosure when the agent has true p-type s but reports t. When $s < t$, $g_t(\cdot)$ stochastically dominates $g_s(\cdot)$ by (1). Since $\theta_{t\omega}$ is increasing in ω, we have for any $\hat{t}$,

$$\nu(t, \hat{t}) > \nu(s, \hat{t}) \quad \text{if } s < t. \tag{31}$$

Since $E_v[\theta_v \mid z_k] = \nu(s_k, s_k)$, the function H can be written in terms of λ_t as:

$$H(s, t) = \lambda_t\, \nu(s, t) + (1 - \lambda_t)\, \nu(t, t).$$

We look for the conditions under which the function H is supermodular, which by Lemma 1 ensures that (Z, f) is implementable. Take any s, $\hat{s}$, t, $\hat{t} \in S$ such that $s < \hat{s}$ and $t < \hat{t}$.

$$\begin{aligned} & H(\hat{s}, \hat{t}) - H(s, \hat{t}) \geq H(\hat{s}, t) - H(s, t) \\ \Leftrightarrow \quad & \left\{\lambda_{\hat{t}}\, \nu(\hat{s}, \hat{t}) + (1 - \lambda_{\hat{t}})\, \nu(\hat{t}, \hat{t})\right\} - \left\{\lambda_{\hat{t}}\, \nu(s, \hat{t}) + (1 - \lambda_{\hat{t}})\, \nu(\hat{t}, \hat{t})\right\} \\ & \geq \left\{\lambda_t\, \nu(\hat{s}, t) + (1 - \lambda_t)\, \nu(t, t)\right\} - \left\{\lambda_t\, \nu(s, t) + (1 - \lambda_t)\, \nu(t, t)\right\}. \end{aligned}$$

Since $\nu(\hat{s}, \hat{t}) - \nu(s, \hat{t}) > 0$ when $\hat{s} > s$ as noted above, H is supermodular if and only if

$$\frac{\lambda_{\hat{t}}}{\lambda_t} \geq \frac{\nu(\hat{s}, t) - \nu(s, t)}{\nu(\hat{s}, \hat{t}) - \nu(s, \hat{t})} \quad \text{if } s < \hat{s} \text{ and } t < \hat{t}. \tag{32}$$

By assumption, θ is submodular so that $\theta_{\hat{t}\omega} - \theta_{t\omega}$ is decreasing in ω when $t < \hat{t}$. By the stochastic dominance (1) of $g_{\hat{s}}$ over g_s, we then have

$$\nu(\hat{s}, \hat{t}) - \nu(\hat{s}, t) = \sum_{\omega} g_{\hat{s}}(\omega)\,(\theta_{\hat{t}\omega} - \theta_{t\omega}) \leq \sum_{\omega} g_s(\omega)\,(\theta_{\hat{t}\omega} - \theta_{t\omega}) = \nu(s, \hat{t}) - \nu(s, t).$$

It follows that the right-hand side of (32) satisfies

$$\frac{\nu(\hat{s}, t) - \nu(s, t)}{\nu(\hat{s}, \hat{t}) - \nu(s, \hat{t})} \geq 1.$$

Hence, (Z, f) is implementable when λ_t is chosen to be increasing in t to satisfy the inequalities in (32). In particular, for $k < \ell$, denote

$$\psi_{k\ell} = \min_{s<\hat{s}} \frac{\nu(\hat{s}, s_k) - \nu(s, s_k)}{\nu(\hat{s}, s_\ell) - \nu(s, s_\ell)} \geq 1.$$

Then (32) holds if for any $\lambda_{s_K} > 0$, $\lambda_{s_1}, \ldots, \lambda_{s_{K-1}}$ satisfy

$$\lambda_{s_k} = \min \frac{\lambda_{s_K}}{\psi_{k_1k_2} \cdots \psi_{k_{m-1}k_m}} \quad \text{for each } k = 1, \ldots, K-1,$$

where minimization is taken over all sequences $(k_1, \ldots, k_m)$ such that $k_1 = k < k_2 < \cdots < k_{m-1} < k_m = K$.

Proposition 9 *Let p, $\succcurlyeq$ and $\eta > 0$ be given, and suppose that the quality function $\theta \in \Theta_{\succcurlyeq,\eta}$ is submodular and defined over $V = S \times \Omega$ with $L = |\Omega| \geq 3$. Then for N defined in (5), there exists $\varepsilon > 0$ such that if θ is ε-linear, there exists an optimal disclosure rule (Z, f) such that*

- *$Z = V \cup \{z_1, \ldots, z_R\}$ for some $R \leq N$ and $z_1, \ldots, z_R \notin V$.*
- *$|\operatorname{supp}(z^r)| = 2$ for every $r = 1, \ldots, R$.*

Exact specification of pooling under optimal disclosure depends sensitively on the parameter values and is not readily available. However, one important implication of Proposition 9 is that the number R of pooling messages is bounded above by N, which is a function of the number K of p-types alone and independent of the number L of l-types. Put differently, even if L becomes very large, the number of pooling messages under the optimal disclosure rule does not increase indefinitely as long as K is fixed.

Although the disclosure rule described in Example 7 is implementable as long as $\lambda_{s_1}, \ldots, \lambda_{s_K}$ satisfy (32), it is *not* optimal when L is large compared with K under the conditions of Proposition 9: To see this, note that the disclosure rule in

Example 7 has at least $(K-1)\times\binom{L}{2}$ pairs of profiles $(v,\hat{v})$ that are pooled with each other.[30] On the other hand, Proposition 9 shows that the number of pairs of profiles that are pooled together is a function of only K. Suppose for example that the p-types are binary ($K=2$) and that the l-type can take three values ($L=3$). In this case, the disclosure rule in Example 7 with $\lambda_{s_2}=1$ sends no pooling message when $s=s_2$, and sends one pooling message when $s=s_1$. This pooling message has support $\{v_{11},v_{12},v_{13}\}$ and hence pools three $(=\binom{3}{2})$ pairs of profiles: $\{v_{11},v_{12}\}$, $\{v_{12},v_{13}\}$, and $\{v_{11},v_{13}\}$. The variable x that corresponds to this rule hence has three strictly positive coordinates. On the other hand, Proposition 9 shows that in such an environment, there exists a unique pair of profiles $(v,\hat{v})$ for which $x^*_{v\hat{v}}>0$. This implies that the disclosure rule in Example 7 is not optimal for a generic specification of (θ,p).[31] As a final remark, N is an upper bound on the number of inequalities for the cyclical monotonicity of H since some of those inequalities may be redundant in some cases. This is most evident when the l-type is binary ($L=2$). As seen in Section 6.2, the number of inequalities for the supermodularity of ϕ is just $K-1$ so that there exist $N-(K-1)$ redundant inequalities.[32]

8 Conclusion

We formulate a model in which an agent's quality is a function of a privately perceptible component (p-type) and a latent component (l-type). The agent reports his p-type to a principal and then takes a test which reveals his l-type. The analysis highlights the intricacy of managing the agent's incentive and minimizing the loss through information disclosure. Full disclosure is optimal when the quality function θ is supermodular, but pooling is required when θ is submodular. When pooling is required, the number of pooling messages is bounded above by the number of inequalities that ensure the implementability of the disclosure rule. Whether the agent's p-type and l-type are complements or substitutes has little bearing on the conclusion.

The novelty of our analysis is the description of a disclosure rule in terms of the mean dual-belief $\bar{\psi}$, which describes the joint distribution of the prior and mean posterior beliefs over the agent's profile $v=(s,\omega)$. The key step is to write both the

[30] Assume that $\lambda_{s_K}=1$ so that the l-type is perfectly revealed when the agent reports the highest p-type s_K.

[31] Under a generic specification of (θ,p), the linear programming problem in terms of x has a unique corner solution.

[32] When $K=3$, for example, $N=5$ and $K-1=2$ so that three inequalities are redundant. The redundant inequalities are those corresponding to: $(k_1,k_2)=(1,3)$, $(k_1,k_2,k_3)=(1,2,3)$, and $(k_1,k_2,k_3)=(1,3,2)$.

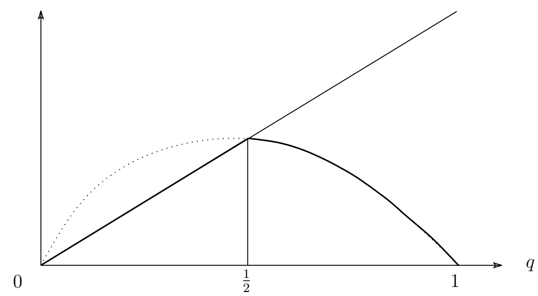

Figure 6: Profile of the minimized loss L^*

The graph depicts L^* along the vertical line segment $\beta = 1$ ($\Leftrightarrow \theta_2 = \theta_3$) in Figure 4.

$$L^* = \begin{cases} \gamma(1-\gamma)q\Delta(\theta_2 - \theta_1) & \text{if } q \leq \frac{1}{2}, \\ \frac{\gamma(1-\gamma)q(1-q)}{\gamma q + (1-\gamma)(1-q)}\,\Delta(\theta_3 - \theta_1) & \text{if } q \geq \frac{1}{2}. \end{cases}$$

principal's objective function and the feasibility constraints as linear transformations of the mean dual-belief. Since implementability of a disclosure rule imposes a restriction on the distribution of posterior beliefs, we cannot apply the standard concavification argument of Bayesian persuasion. To see this point, consider the 2×2 environment of Section 6.1. Full disclosure is trivially implementable when the agent has the low p-type with probability one, or when he has the high p-type with probability one. If both p-types occur with positive probability, however, full disclosure is no longer implementable if θ is submodular. Figure 6 illustrates this point by depicting the quadratic loss under the optimal disclosure rule in the 2×2 example of Section 6.1 corresponding to Figure 4. Since the principal's objective is to *minimize* the loss, cancavification in the current context would imply a convex function. This however is not the case: Full disclosure is optimal and hence entails no loss at both ends ($q = \Pr(s_1) \in \{0, 1\}$) where the agent's p-type is known with probability one, but not implementable at interior points where submodularity creates an incentive issue.

It is conceivable in some situations that the agent may be interested in enhancing his quality before interacting with the principal. For example, a student may make effort to acquire software skills before attending college, or a company may invest in the improvement of its product before applying for a certification program. In Appendix C (online), we identify an optimal mechanism in the 2×2 environment when the agent has a binary ex ante action choice which determines the probability distribution of his p-type s. In line with the intuition developed in the text, the optimal disclosure rule is such that the number of pooling messages equals the number of inequalities that ensure implementability, which in this case corresponds to incentive compatibility for truthful reporting and incentive to exert costly effort that stochastically enhances the p-type.

The agent's incentive compatibility conditions may be relevant in a setting where the principal has a different objective than minimizing the quadratic loss. In Appendix D (online), we illustrate the implications of incentive compatibility in a common framework of persuasion where the agent receives a fixed compensation if and only if his expected quality is at or above a certain threshold and the principal's objective is to maximize the probability that the agent receives compensation. In the simple 2×2 environment, we use the dual-belief construction developed in Appendix A to show that implementability places a direct restriction on the posterior belief that follows the principal's recommendation to hire the agent. We argue that such a requirement typically creates a loss in the principal's payoff compared to when the agent's p-type is directly observable.

This paper focuses on a model without any formal contract with monetary transfer. Alternatively, we may assume that the principal sells information to the market and also compensates the agent for his participation in the mechanism.[33] Examining such a framework, encompassing formal contracts between the agent and principal and between the market and the principal, along with informal contracts between the agent and market, could be an interesting avenue for future research.

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Appendix A Disclosure rule and dual-belief distributions

The distribution of posterior beliefs $\hat{\tau} \in \Delta(\Delta V)$ is *consistent* if

$$\sum_{\zeta} \hat{\tau}(\zeta)\,\zeta(\hat{v}) = p(\hat{v}) \quad \text{for every } \hat{v} \in V. \tag{33}$$

As is well-known, there is a one-to-one correspondence between a consistent distribution of posterior beliefs and a disclosure rule. Given a probability distribution ζ over V and profile $\tilde{v} \in V$, a *dual-belief* $\psi_{v,\zeta}$ is a probability distribution over pairs of profiles $(v', v'') \in V^2$ such that

$$\psi_{v,\zeta}(v', v'') = \begin{cases} \zeta(v'') & \text{if } v' = v, \\ 0 & \text{otherwise.} \end{cases}$$

That is, when $\psi_{v,\zeta}$ is identified as a $V \times V$ matrix, then it has a unique non-zero row in row v, and this row equals ζ. The interpretation is that given a disclosure rule, each dual-belief represents a realization of a pair of a profile and a posterior belief. Denote by $\mathcal{D} = V \times \Delta V$ the set of all dual-beliefs and consider the set of probability distributions τ over $\mathcal{D}$ as follows:

$$\begin{aligned} \mathcal{T} = \big\{\tau \in \Delta\mathcal{D} : &\text{There exists a consistent } \hat{\tau} \in \Delta(\Delta V) \text{ such that} \\ &\tau(\psi_{v,\zeta}) = \hat{\tau}(\zeta)\,\zeta(v) \text{ for every } (v,\zeta)\big\}. \end{aligned} \tag{34}$$

We say that $\tau \in \mathcal{T}$ is a *consistent dual-belief distribution.* The following proposition establishes that a consistent dual-belief distribution corresponds one-to-one to a disclosure rule.

Proposition 10 *1. Suppose that (Z, f) is a disclosure rule such that $\sum_v p_v\, f(z \mid v) > 0$ for every $z \in Z$. Let $\zeta_z \in \Delta V$ be*

$$\zeta_z(\hat{v}) = \frac{p_{\hat{v}}\, f(z \mid \hat{v})}{\sum_v p_v\, f(z \mid v)} \quad \text{for every } z \in Z \text{ and } \hat{v} \in V.$$

If we define $\tau_f \in \mathcal{D}$ by

$$\tau_f(\psi_{v,\zeta}) = \begin{cases} p_v\, f(z \mid v) & \text{if } \zeta = \zeta_z, \\ 0 & \text{otherwise,} \end{cases} \tag{35}$$

then $\tau_f \in \mathcal{T}$.

2. Suppose that $\tau \in \mathcal{T}$. If we let $Z = \Delta V$ and $f : V \to \Delta Z$ be defined by

$$f(\zeta \mid v) = \frac{1}{p_v}\, \tau(\psi_{\tilde{v},\zeta}) \quad \text{for every } \zeta \in \Delta V \text{ and } v \in V,$$

then (Z, f) is a disclosure rule.

Given the disclosure rule (Z, f), (35) shows that $\tau_f(\psi_{v,\zeta_z})$ is the joint probability of (v, z). Hence, if (Z, f) is specified so that there is a one-to-one correspondence between message z and the receiver's action a, then τ_f is the joint distribution of state v and action a as studied by Kolotilin et al. (2022).

Proof of Proposition 10.
1. Define

$$\hat{\tau}(\zeta) = \begin{cases} \sum_v p_v\, f(z \mid v) & \text{if } \zeta = \zeta_z \text{ for some } z \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\hat{\tau}$ is consistent and also satisfies

$$\tau_f(\psi_{v,\zeta}) = p_{\hat{v}}\, f(z \mid \hat{v}) = \zeta_z(\hat{v}) \sum_v p_v\, f(z \mid v) = \hat{\tau}(\zeta_z)\, \zeta_z(\hat{v}).$$

It follows that $\tau_f \in \mathcal{T}$.

2. (Z, f) is a disclosure rule since for any $v \in V$,

$$\sum_{\zeta \in \Delta V} f(\zeta \mid v) = \sum_{\zeta \in \Delta V} \frac{1}{p_v}\, \tau(\psi_{v,\zeta}) = \frac{1}{p_v} \sum_{\zeta \in \Delta V} \hat{\tau}(\zeta)\, \zeta(v) = \frac{1}{p_v}\, p_v = 1,$$

where the third equality follows since $\hat{\tau}$ is consistent. ■

Given $\tau \in \mathcal{T}$, the *mean dual-belief* $\bar{\psi}$ is a probability distribution over V^2 defined by

$$\bar{\psi}(v', v'') = \sum_{v,\zeta} \tau(\psi_{v,\zeta})\, \psi_{v,\zeta}(v', v'') \quad \text{for every } (v', v'') \in V^2. \tag{36}$$

Since $\psi_{v,\zeta}(v', v'') = 0$ when $v' \neq v$, we can rewrite $\bar{\psi}$ as:

$$\bar{\psi}(v', v'') = \sum_{\zeta} \tau(\psi_{v',\zeta})\, \psi_{v',\zeta}(v, v'') = \sum_{\zeta} \hat{\tau}(\zeta)\, \zeta(v')\, \zeta(v''). \tag{37}$$

Hence, $\bar{\psi}(v, \hat{v})$ is the mean of the product of posterior beliefs $\zeta(v)$ and $\zeta(\hat{v})$. It then follows that $\bar{\psi}$ is symmetric ($\bar{\psi}(v, \hat{v}) = \bar{\psi}(\hat{v}, v)$), and that the marginal distribution of $\bar{\psi}$ equals the prior p: Since $\hat{\tau}$ is consistent, for any $\hat{v} \in V$,

$$\sum_{v} \bar{\psi}(v, \hat{v}) = \sum_{v} \sum_{\zeta} \hat{\tau}(\zeta)\, \zeta(v)\, \zeta(\hat{v}) = \sum_{\zeta} \hat{\tau}(\zeta)\, \zeta(\hat{v}) = p(\hat{v}).$$

Appendix B Proofs

Proof of Lemma 1. When $n = 2$, (4) holds under supermodularity since

$$H(s_1, s_1) - H(s_1, s_2) + H(s_2, s_2) - H(s_2, s_1) \geq 0.$$

As an induction hypothesis, suppose that supermodularity implies (4) for $n = m-1$. Suppose that $n = m$, and assume without loss of generality that $k_0 = k_m > \max\{k_1, \ldots, k_{m-1}\}$. Then,

$$\begin{aligned}
&\sum_{i=1}^{m} \{H(s_{k_i}, s_{k_i}) - H(s_{k_i}, s_{k_{i-1}})\} \\
&= \sum_{i=1}^{m-1} \{H(s_{k_i}, s_{k_i}) - H(s_{k_i}, s_{k_{i-1}})\} + H(s_{k_m}, s_{k_m}) - H(s_{k_m}, s_{k_{m-1}}) \\
&\geq \sum_{i=1}^{m-1} \{H(s_{k_i}, s_{k_i}) - H(s_{k_i}, s_{k_{i-1}})\} + H(s_{k_1}, s_{k_m}) - H(s_{k_1}, s_{k_{m-1}}) \\
&= \sum_{i=2}^{m-1} \{H(s_{k_i}, s_{k_i}) - H(s_{k_i}, s_{k_{i-1}})\} + H(s_{k_1}, s_{k_1}) - H(s_{k_1}, s_{k_m}) \\
&\qquad\qquad + H(s_{k_1}, s_{k_m}) - H(s_{k_1}, s_{k_{m-1}}) \\
&= H(s_{k_1}, s_{k_1}) - H(s_{k_1}, s_{k_{m-1}}) + \sum_{i=2}^{m-1} \{H(s_{k_i}, s_{k_i}) - H(s_{k_i}, s_{k_{i-1}})\} \\
&\geq 0,
\end{aligned}$$

where the first inequality follows from the supermodularity of H and the second inequality from the induction hypothesis. ■

Proof of Proposition 4. Let (Z, f) be the full disclosure rule so that $\phi(s,\omega) = \theta(s,\omega)$ for every (s,ω). We then have

$$H(s,t) = \sum_{\omega\in\Omega} g_s(\omega)\,\theta(t,\omega).$$

Take any s, $\hat{s}$, t, and $\hat{t} \in S$ such that $s < \hat{s}$ and $t < \hat{t}$. Since θ is supermodular, $\theta(\hat{t},\omega) - \theta(t,\omega)$ is an increasing function of ω, and since $g_{\hat{s}}(\omega)$ (first-order) stochastically dominates $g_s(\omega)$ by (1),

$$\begin{aligned} H(\hat{s},\hat{t}) - H(\hat{s},t) &= \sum_{\omega\in\Omega} g_{\hat{s}}(\omega)\,\{\theta(\hat{t},\omega) - \theta(t,\omega)\} \\ &\geq \sum_{\omega\in\Omega} g_s(\omega)\,\{\theta(\hat{t},\omega) - \theta(t,\omega)\} \\ &= H(s,\hat{t}) - H(s,t). \end{aligned}$$

It follows that H is supermodular, and hence is cyclically monotone by Lemma 1. It follows that (Z, f) is implementable by Proposition 2. ■

Proof of Lemma 3. It suffices to show that (4) $\Leftrightarrow$ (6).
(4) $\Rightarrow$ (6). When $m = 2$, $s_{k_1} = s$ and $s_{k_2} = t$, (4) is written as:

$$H(s,t) - H(s,s) \leq H(t,t) - H(t,s),$$

which is equivalent to

$$\sum_{\omega} g_s(\omega)\,\{\phi(t,\omega) - \phi(s,\omega)\} \leq \sum_{\omega} g_t(\omega)\,\{\phi(t,\omega) - \phi(s,\omega)\}. \tag{38}$$

Substituting $g_s(\omega_1) = 1 - g_s(\omega_2)$ and $g_t(\omega_1) = 1 - g_t(\omega_2)$ and then simplifying, we see that (38) is equivalent to

$$\{g_t(\omega_2) - g_s(\omega_2)\}\left\{\phi(t,\omega_2) - \phi(s,\omega_2) - \phi(t,\omega_1) + \phi(s,\omega_1)\right\} \geq 0.$$

When $s < t$, $g_t(\omega_2) - g_s(\omega_2) > 0$ by stochastic dominance (1) so that (6) holds.
(6) $\Rightarrow$ (4). Take any $s, \hat{s}, t, \hat{t} \in S$ such that $s < \hat{s}$ and $t < \hat{t}$. By (6), $\phi(\hat{t},\omega) - \phi(t,\omega)$ is an increasing function of ω. Since $g_{\hat{s}}$ stochastically dominates g_s by (1), we have

$$\sum_{\omega} g_s(\omega)\,\{\phi(\hat{t},\omega) - \phi(t,\omega)\} \leq \sum_{\omega} g_{\hat{s}}(\omega)\,\{\phi(\hat{t},\omega) - \phi(t,\omega)\}.$$

By the definition of H, this is equivalent to

$$H(s,\hat{t}) - H(s,t) \le H(\hat{s},\hat{t}) - H(\hat{s},t),$$

which shows that H is supermodular. It then follows from Lemma 1 that H is cyclically monotone (4).

We next show (7) $\Rightarrow$ (6) since the implication (6) $\Rightarrow$ (7) is clear. Take any s_m, $s_n \in S$ with $m < n$. Since (7) holds for $k = m, \ldots, n-1$, we have

$$\begin{aligned} \phi(s_n,\omega_2) - \phi(s_n,\omega_1) &\ge \phi(s_{n-1},\omega_2) - \phi(s_{n-1},\omega_1) \\ &\ge \cdots \\ &\ge \phi(s_{m+1},\omega_2) - \phi(s_{m+1},\omega_1) \\ &\ge \phi(s_m,\omega_2) - \phi(s_m,\omega_1), \end{aligned}$$

which implies (6). ■

Proof of Lemma 5. When ϕ, H and $\mathcal{L}$ are respectively given by (17), (18) and (19), we can rewrite them as functions of x in (22) as follows:

$$\phi(s,\omega) = \theta_{s\omega} + \sum_v p_v\, x_{v,s\omega}\, (-1)^{\mathbf{1}_{\{v \prec s\omega\}}}, \tag{39}$$

$$H(s,t) = \sum_\omega g_s(\omega) \Big[\theta_{t\omega} + \sum_v p_v\, x_{v,t\omega}\, (-1)^{\mathbf{1}_{\{v\prec t\omega\}}}\Big], \tag{40}$$

$$\mathcal{L}(\Gamma) = \sum_{\{(v,\hat{v}):\, v\prec\hat{v}\}} p_v p_{\hat{v}}\, x_{v\hat{v}}\, |\theta_{\hat{v}} - \theta_v|, \tag{41}$$

where $\mathbf{1}_E$ is the indicator function of event E so that $(-1)^{\mathbf{1}_{\{v\prec s\omega\}}} = \begin{cases} -1 & \text{if } \theta_v < \theta_{s\omega}, \\ 1 & \text{if } \theta_v \ge \theta_{s\omega}. \end{cases}$

1. Ex post compensation function $\phi(v)$: (17) follows since

   $$\phi(v) = \sum_z f(z \mid v) \big(\sum_{\hat{v}} \zeta_z(\hat{v})\, \theta_{\hat{v}}\big) = \sum_{\hat{v}} \theta_{\hat{v}} \sum_z f(z \mid v)\, \zeta_z(\hat{v}) = \frac{1}{p_v} \sum_{\hat{v}} \theta_{\hat{v}}\, \bar{\psi}_f(v,\hat{v}).$$

   Transforming this further, we obtain

   $$\begin{aligned} \phi(v) &= \frac{\theta_v}{p_v}\, \bar{\psi}_f(v,v) + \frac{1}{p_v} \sum_{\hat{v}\neq v} \theta_{\hat{v}}\, \bar{\psi}_f(v,\hat{v}) \\ &= \frac{\theta_v}{p_v} \big\{p_v - \sum_{\hat{v}\neq v} \bar{\psi}_f(v,\hat{v})\big\} + \frac{1}{p_v} \sum_{\hat{v}\neq v} \theta_{\hat{v}}\, \bar{\psi}_f(v,\hat{v}) \\ &= \theta_v\ + \frac{1}{p_v} \sum_{\hat{v}\neq v} \bar{\psi}_f(v,\hat{v})\, (\theta_{\hat{v}} - \theta_v). \end{aligned}$$

   Substitution of the definition of x yields (39).

2. Interim compensation function $H(s,t)$: Substitution of (17) yields

$$H(s,t) = \sum_{\omega} g_s(\omega)\, \phi(t,\omega) = \sum_{\omega} \frac{g_s(\omega)}{p(t,\omega)} \sum_{\hat{v}} \theta_{\hat{v}}\, \bar{\psi}_f((t,\omega),\hat{v}).$$

(40) likewise follows from the substitution of (39) into the above expression of H.

3. Quadratic loss function $\mathcal{L}$: If we write $\bar{f}(z) = \sum_v p_v\, f(z \mid v)$ for every $z \in Z$, then $\mu_z = \sum_{\hat{v}} \zeta_z(\hat{v})\, \theta_{\hat{v}} = \frac{1}{\bar{f}(z)} \sum_v p_v f(z \mid v)\theta_v$. Since

$$\sum_z \mu_z\, f(z \mid v) = \phi(v) = \frac{1}{p_v} \sum_{\hat{v}} \theta_{\hat{v}}\, \bar{\psi}_f(v,\hat{v}),$$

we obtain

$$\begin{aligned}
\sum_z \bar{f}(z)\, \mu_z^2 &= \sum_z \bar{f}(z)\, \mu_z \frac{1}{\bar{f}(z)} \sum_v p_v f(z \mid v)\theta_v \\
&= \sum_z \mu_z \sum_v p_v f(z \mid v)\theta_v \\
&= \sum_v p_v\theta_v \sum_z \mu_z\, f(z \mid v) \\
&= \sum_{v,\hat{v}} \bar{\psi}_f(v,\hat{v})\, \theta_v\theta_{\hat{v}}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\mathcal{L}(\Gamma) &= \sum_{v,z} p_v f(z \mid v)\, (\theta_v - \mu_z)^2 = \sum_{v,z} p_v f(z \mid v)\, \big(\theta_v^2 - 2\theta_v\, \mu_z + \mu_z^2\big) \\
&= \sum_v p_v\theta_v^2 - 2\sum_v p_v\theta_v \sum_{\hat{v}} \frac{\theta_{\hat{v}}}{p_v}\, \bar{\psi}_f(v,\hat{v}) + \sum_z \bar{f}(z)\, \mu_z^2 \\
&= \sum_{v,\hat{v}} \bar{\psi}_f(v,\hat{v})\, \theta_v \big(\theta_v - \theta_{\hat{v}}\big) = \sum_{v \prec \hat{v}} \bar{\psi}_f(v,\hat{v})\, (\theta_{\hat{v}} - \theta_v)^2,
\end{aligned}$$

where the last equality holds because of the symmetry of $\bar{\psi}_f$.

■

Proof of Proposition 6. Let (Z,f) be any disclosure rule as described in (20). By Lemma 3, (Z,f) is implementable if and only if ϕ is supermodular:

$$\phi_1 + \phi_4 - \phi_2 - \phi_3 \geq 0. \tag{42}$$

Using Lemma 5, we can rewrite (42) as

$$\sum_v p_v x_{v1} (-1)^{\mathbf{1}_{\{v \prec 1\}}} + \sum_v p_v x_{v4} (-1)^{\mathbf{1}_{\{v \prec 4\}}}$$
$$- \sum_v p_v x_{v2} (-1)^{\mathbf{1}_{\{v \prec 2\}}} - \sum_v p_v x_{v3} (-1)^{\mathbf{1}_{\{v \prec 3\}}} \geq \Delta,$$

where Δ is as defined in (10) and $\Delta > 0$ when θ is submodular. Collecting terms while noting $1 \prec 2$, $1 \prec 3$, $1 \prec 4$, $2 \prec 4$, and $3 \prec 4$, we obtain

$$\begin{aligned} &x_{12} (p_1 + p_2) + x_{13} (p_1 + p_3) - x_{24} (p_2 + p_4) - x_{34} (p_3 + p_4) \\ &+ x_{14} (p_4 - p_1) - (-1)^{\mathbf{1}_{\{2 \prec 3\}}} x_{23} (p_2 - p_3) \geq \Delta. \end{aligned} \tag{43}$$

In other words, (43) is an explicit form of the implementability condition in problem (29). When each x_{mn} can take any non-negative values, there exists an optimal solution x^* such that $x^*_{mn} > 0$ for a unique pair (m, n) $(m \prec n)$ and $x^*_{mn} = 0$ for any other pair. Furthermore, $x^*_{24} = x^*_{34} = 0$ since their coefficients in (43) are unambiguously negative. It follows that the optimal solution x^* to this linear problem and the corresponding optimum L^* are given by one of the following:

i) $x^*_{12} = \frac{\Delta}{p_1+p_2} \Rightarrow L^*_{12} = \frac{p_1 p_2}{p_1+p_2} \Delta (\theta_2 - \theta_1)$.
ii) $x^*_{13} = \frac{\Delta}{p_1+p_3} \Rightarrow L^*_{13} = \frac{p_1 p_3}{p_1+p_3} \Delta (\theta_3 - \theta_1)$.
iii) $x^*_{23} = \frac{\Delta}{|p_2-p_3|} \Rightarrow L^*_{23} = \frac{p_2 p_3}{p_2-p_3} \Delta (\theta_3 - \theta_2)$ if $(p_2 - p_3)(\theta_3 - \theta_2) > 0$.
iv) $x^*_{14} = \frac{\Delta}{p_4-p_1} \Rightarrow L^*_{14} = \frac{p_1 p_4}{p_4-p_1} \Delta (\theta_4 - \theta_1)$ if $p_4 > p_1$.

Among these, we see that case (iv) is dominated by case (i): $L^*_{14} < L^*_{12}$ for any (θ, p).[34] We proceed by separately considering conditions (25)-(27).

1. (θ, p) satisfies (25): Since case (iii) is irrelevant in this case, either case (i) or case (ii) is optimal. We note that $L^*_{12} \leq L^*_{13}$ if and only if

$$\frac{p_2}{p_1 + p_2} (\theta_2 - \theta_1) \leq \frac{p_3}{p_1 + p_3} (\theta_3 - \theta_1) \quad \Leftrightarrow \quad \frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} \leq \frac{p_3(p_1 + p_2)}{p_2(p_1 + p_3)}.$$

2. (θ, p) satisfies (26): $(p_2 - p_3) \left(\frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} - \frac{p_3(p_1+p_2)}{p_2(p_1+p_3)} \right) \leq 0$.[35] We first show that (26) is equivalent to both

$$L^*_{12} \leq L^*_{23} \quad \text{and} \quad L^*_{13} \leq L^*_{23}.$$

[34] When $p_4 > p_1$, $\frac{p_1 p_2}{p_1+p_2} \Delta (\theta_2 - \theta_1) < \frac{p_1 p_4}{p_4 - p_1} \Delta (\theta_4 - \theta_1)$.
[35] Note that (26) implies $(p_2 - p_3)(\theta_3 - \theta_2) > 0$.

Suppose that $p_2 - p_3 > 0$. From cases (i) and (iii), we see that $L_{12}^* \leq L_{23}^*$ if and only if

$$\begin{aligned}
& \frac{p_3}{p_2 - p_3}(\theta_3 - \theta_2) \geq \frac{p_1}{p_1 + p_2}(\theta_2 - \theta_1) \\
\Leftrightarrow \quad & p_3(p_1 + p_2)(\theta_3 - \theta_2) \geq p_1(p_2 - p_3)(\theta_2 - \theta_1) \\
\Leftrightarrow \quad & p_3(p_1 + p_2)(\theta_3 - \theta_1) \geq p_2(p_1 + p_3)(\theta_2 - \theta_1) \\
\Leftrightarrow \quad & \frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} \leq \frac{p_3(p_1 + p_2)}{p_2(p_1 + p_3)}.
\end{aligned}$$

Likewise, from cases (ii) and (iii), we see that $L_{13}^* \leq L_{23}^*$ if and only if

$$\begin{aligned}
& \frac{p_2}{p_2 - p_3}(\theta_3 - \theta_2) \geq \frac{p_1}{p_1 + p_3}(\theta_3 - \theta_1) \\
\Leftrightarrow \quad & p_2(p_1 + p_3)(\theta_3 - \theta_2) \geq p_1(p_2 - p_3)(\theta_3 - \theta_1) \\
\Leftrightarrow \quad & p_3(p_1 + p_2)(\theta_3 - \theta_1) \geq p_2(p_1 + p_3)(\theta_2 - \theta_1) \\
\Leftrightarrow \quad & \frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} \leq \frac{p_3(p_1 + p_2)}{p_2(p_1 + p_3)}.
\end{aligned}$$

When $p_2 - p_3 < 0$, we can likewise show that both $L_{12}^* \leq L_{23}^*$ and $L_{13}^* \leq L_{23}^*$ are equivalent to

$$\frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} \geq \frac{p_3(p_1 + p_2)}{p_2(p_1 + p_3)}.$$

When $p_2 - p_3 = 0$, (26) always holds and so do $L_{12}^* \leq L_{23}^*$ and $L_{13}^* \leq L_{23}^*$.
We next show that in each of case (i) and case (ii), there exists $\alpha = (\alpha_m^r)_{m,r}$ that generates x^*. In case (i), let

$$R = 1, \quad \text{and} \quad \alpha_m^1 = \begin{cases} \frac{\Delta}{\theta_2 - \theta_1} & \text{if } m = 1 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

α then satisfies $\alpha_1^1 = \alpha_2^1 \in (0, 1)$ since $\frac{\Delta}{\theta_2 - \theta_1} < 1$, and generates from (22) x^* in case (i): $x_{mn}^* = 0$ for any $(m, n) \neq (1, 2)$ and

$$x_{12}^* = \frac{\Delta}{p_1 + p_2}.$$

In case (ii), let

$$R = 1, \quad \text{and} \quad \alpha_m^1 = \begin{cases} \frac{\Delta}{\theta_3 - \theta_1} & \text{if } m = 1 \text{ or } 3, \\ 0 & \text{otherwise.} \end{cases}$$

α then satisfies $\alpha_1^1 = \alpha_3^1 \in (0,1)$ since $\frac{\Delta}{\theta_3-\theta_1} < 1$, and generates from (22) x^* in case (ii): $x^*_{mn} = 0$ for any $(m,n) \neq (1,3)$ and

$$x^*_{13} = \frac{\Delta}{p_1 + p_3}.$$

3. (θ, p) satisfies (27) but violates (26).
   Since (27) implies $(p_2 - p_3)(\theta_3 - \theta_2) > 0$, case (iii) is relevant, and indeed optimal since the violation of (26) is equivalent to $L^*_{23} < L^*_{12}$ and $L^*_{23} < L^*_{13}$ as seen above. Let

$$R = 1, \quad \text{and} \quad \alpha_m^1 = \begin{cases} \frac{(p_2+p_3)\,\Delta}{(p_2-p_3)(\theta_3-\theta_2)} & \text{if } m = 2 \text{ or } 3, \\ 0 & \text{otherwise.} \end{cases}$$

   α then satisfies $\alpha_2^1 = \alpha_3^1 \in (0,1)$ by (27), and generates x^* in case (iii): $x^*_{mn} = 0$ for any $(m,n) \neq (2,3)$, and

$$x^*_{23} = \frac{\alpha_2^1 \alpha_3^1}{p_2 \alpha_2^1 + p_3 \alpha_3^1} \, |\theta_3 - \theta_2| = \frac{\Delta}{|p_2 - p_3|}.$$

   It follows that $L^* = L^*_{23}$.

This completes the proof. ■

Proof of Corollary 7. The conclusion is immediate if (θ, p) satisfies the sufficient condition (26) of Proposition 6. Suppose then that (θ, p) violates (26). This implies that $(p_2 - p_3)(\theta_3 - \theta_2) > 0$. Since θ is ε-linear, we have

$$\Delta = 2\Big(\frac{\theta_3 - \theta_1}{2} - \frac{\theta_4 - \theta_2}{2}\Big) \leq 2\big((h + \varepsilon) - (h - \varepsilon)\big) = 4\varepsilon.$$

Hence, for ε as given, $\theta \in \Theta_{\not\approx,\eta}$ implies

$$\Delta \leq 4\varepsilon \leq \eta \frac{|p_2 - p_3|}{p_2 + p_3} < \frac{(p_2 - p_3)(\theta_3 - \theta_2)}{p_2 + p_3},$$

which again shows that the sufficient condition (27) of Proposition 6 holds. ■

Proof of Proposition 8. Since θ is assumed to be submodular,

$$\Delta_k \equiv \theta_{k2} + \theta_{k+1,1} - \theta_{k1} - \theta_{k+1,2} > 0 \quad \text{for } k = 1, \ldots, K-1.$$

Note that ϕ is supermodular if and only if

$$\phi_{k1} + \phi_{k+1,2} - \phi_{k2} - \phi_{k+1,1} \geq 0 \quad \text{for } k = 1, \ldots, K-1. \tag{44}$$

Using (39), we can rewrite (44) as

$$\sum_v p_v \Big[x_{v,s_k\omega_1}(-1)^{\mathbf{1}_{\{v\prec k1\}}} + x_{v,s_{k+1}\omega_2}(-1)^{\mathbf{1}_{\{v\prec k+1,2\}}} \\ -x_{v,s_k\omega_2}(-1)^{\mathbf{1}_{\{v\prec k2\}}} - x_{v,s_{k+1}\omega_1}(-1)^{\mathbf{1}_{\{v\prec k+1,1\}}}\Big] \geq \Delta_k \\ \text{for } k=1,\ldots,K-1. \tag{45}$$

When θ is ε-linear, $\Delta_k < 4\varepsilon$ so that $\Delta_k \to 0$ for each k as $\varepsilon \to 0$. We proceed in the following steps. In step 1, we consider minimization of $\mathcal{L}$ with respect to $x \geq 0$ subject to (45), and show that there exists a solution x^* which has at most $K-1$ positive entries. In step 2, we show that when Δ is small, the solution x^* is close to 0. In step 3, we show that when x^* is small, there exists α that generates it, and corresponds to $K-1$ imperfect messages each with support consisting of two profiles.

1. Consider minimizing (41) with respect to $x = (x_{v\hat{v}})_{v\prec\hat{v}}$ subject to (45) as well as $x_{v\hat{v}} \geq 0$. The set of solutions is non-empty since x that corresponds to the implementable disclosure rule in Example 6 satisfies the feasibility constraints. Let $q = \binom{2K}{2}$ denote the dimension of $x = (x_{v\hat{v}})_{v\prec\hat{v}}$. Since (45) involves $K-1$ inequalities, if we denote by A the $(K-1)\times q$ matrix of coefficients on x, then (45) can be expressed in matrix form as

$$Ax \geq \Delta \equiv \begin{bmatrix} \Delta_1 \\ \vdots \\ \Delta_{K-1} \end{bmatrix}.$$

   The optimization problem with respect to x (corresponding to (29) in the 2×2 model) is hence written as:

$$\min \sum_{\{(v,\hat{v}):\, v\prec\hat{v}\}} p_v p_{\hat{v}}\, x_{v\hat{v}}\, |\theta_{\hat{v}} - \theta_v| \\ \text{subject to: } x \in P \equiv \{x : \, Ax \geq \Delta,\, x \geq 0\}. \tag{46}$$

   Since the objective function is also linear in x, there exists a solution x^* to (46) which is an extreme point of the polyhedron P. Let $J = \{j :\, j = (v,\hat{v}),\, x^*_{v\hat{v}} > 0\}$ be the indices of strictly positive entries of x^*.
   We will show that $|J| \leq K-1$. Suppose to the contrary that $|J| \geq K$, and consider the collection $(e_j)_{j\in J}$, where e_j is the jth unit vector, which has 1 in the jth entry and zero in all other entries. Denote also by $(\zeta_i)_{i=1}^d$ the base of the null space of A: $\{x :\, Ax = 0\}$. Since the dimension d of this null space satisfies $d = q - \text{rank}(A) \geq q - (K-1)$, the collection of $d + |J| \geq$

$q - K + 1 + K = q + 1$ vectors $((e_j)_{j \in J}, (\zeta_i)_{i=1}^d)$ is linearly dependent. There then exist $(\lambda_i)_{i=1}^d$ and $(\mu_j)_{j \in J}$ such that $\sum_i \lambda_i \zeta_i + \sum_j \mu_j e_j = 0$, where λ_i's are not all equal to zero, and μ_j's are not all equal to zero. For $\kappa > 0$, consider $\hat{x} \equiv x^* + \kappa \sum_i \lambda_i \zeta_i$ and $\tilde{x} \equiv x^* - \kappa \sum_i \lambda_i \zeta_i$. Since $A\zeta_1 = \cdots = A\zeta_d = 0$,

$$A\hat{x} = A\tilde{x} = Ax^*.$$

Furthermore, since $\sum_i \lambda_i \zeta_i = -\sum_j \mu_j e_j$, if $x^*_{v\hat{v}} = 0$ for any $(v, \hat{v})$ (*i.e.*, $j = (v, \hat{v}) \notin J$), then $\hat{x}_{v\hat{v}} = \tilde{x}_{v\hat{v}} = 0$ as well. It follows that both $\hat{x}$ and $\tilde{x}$ belong to the polyhedron P provided that κ is sufficiently small. Since $x^* = (\hat{x} + \tilde{x})/2$, this contradicts the fact that x^* is an extreme point of P.

2. When $\Delta = 0$, it is clear that (46) has a solution $x = 0$. We show that when Δ is small, any solution to (46) is close to zero using the theorem of the maximum. For this, take $B > 0$ large enough and consider the following maximization problem:

$$\max_x \,(-1) \sum_{\{(v,\hat{v}):\, v \prec \hat{v}\}} p_v p_{\hat{v}}\, x_{v\hat{v}} \,|\theta_{\hat{v}} - \theta_v| \quad \text{subject to } x \in \Lambda(\delta), \tag{47}$$

where

$$\Lambda(\delta) = \{x = (x_{v\hat{v}})_{v \prec \hat{v}} :\, Ax \geq \delta,\; 0 \leq x \leq (B, \ldots, B)\}.$$

The objective function is linear in x and hence continuous. Note also that the matrix A in the constraints is a function only of p and $\succcurlyeq$, and is independent of the choice of $\theta \in \Theta_{\succcurlyeq,\eta}$.

The correspondence $\Lambda :\, \boldsymbol{R}_+^{K-1} \to \boldsymbol{R}^q$ is continuous at $\delta = 0$ and compact-valued: To see that it is upper hemi-continuous at $\delta = 0$, note that for any open set $G \supset \Lambda(0)$ and any $\delta \geq 0$, $\Lambda(0) \supset \Lambda(\delta)$ so that there exists a neighborhood $U \subset \boldsymbol{R}_+^{K-1}$ of $\delta = 0$ such that $\delta \in U$ implies $\Lambda(\delta) \subset G$. To see that Λ is lower hemi-continuous at $\delta = 0$, take any open set $G \subset \boldsymbol{R}^q$ such that $G \cap \Lambda(0) \neq \emptyset$. Let x^0 be an element of this intersection, and $\bar{x} \geq 0$ be the value of x corresponding to the disclosure rule in Example 6 for some fixed $\Delta = \bar{\Delta} \gg 0$ so that $A\bar{x} \geq \bar{\Delta}$. Take $\varepsilon > 0$ small enough so that $\varepsilon\bar{x} + (1 - \varepsilon)x^0 \in G$, and take $U = [0, \varepsilon\bar{\Delta}) \subset \boldsymbol{R}_+^{K-1}$ as a neighborhood of $0 \in \boldsymbol{R}_+^{K-1}$. Then for any $\delta \in U$, we have

$$A\big(\varepsilon\bar{x} + (1 - \varepsilon)x^0\big) \geq \varepsilon A\bar{x} \geq \varepsilon\bar{\Delta} \gg \delta,$$

and hence $\varepsilon\bar{x} + (1 - \varepsilon)x^0 \in \Lambda(\delta)$. It follows that $\delta \in U$ implies that $G \cap \Lambda(\delta) \neq \emptyset$, showing that Λ is lower hemi-continuous at $\delta = 0$.

We note that the original optimization problem is equivalent to (47) for $\delta = \Delta$ since the upper bound B on $x_{v\hat{v}}$ can be ignored if B is taken large enough. Define $G^*(\delta)$ to be the set of solutions to (47) for each $\delta \geq 0$. Note that $G^*(0) = \{0\}$ since $x = 0$ is the unique solution to (47) when $\delta = 0$. If we take an open ball around 0 of radius $\frac{\eta}{K-1}$, then $G^*(0) = \{0\} \subset O$. Since the correspondence $G^* : \boldsymbol{R}_+^{K-1} \to \boldsymbol{R}^q$ is upper hemi-continuous by Berge's theorem of the maximum, there exists a neighborhood $U \subset \boldsymbol{R}_+^{K-1}$ of $\delta = 0$ such that $\delta \in U$ implies $G(\delta) \subset O$, or equivalently, $\delta \in U$ implies $x^*_{v\hat{v}}(\delta) < \frac{\eta}{K-1}$ for every $(v, \hat{v})$ with $v \prec \hat{v}$.

3. Take $\delta > 0$ as above and write $x^* \equiv x^*(\delta)$. Let J be the set of indices of non-zero entries of x^*. By Step 1, $|J| \leq K - 1$. We show that when $x^*_{v\hat{v}} < \frac{\eta}{K-1}$ for every $v \prec \hat{v}$, there exist the set of imperfect messages $\{z_1, \dots, z_R\}$ and their probabilities $\alpha = (\alpha_v^r)_{v,r}$ that replicate x^*. First, let $R = |J|$ and define

$$\{z_1, \dots, z_R\} = \{v\hat{v} : v \prec \hat{v} \text{ and } x^*_{v\hat{v}} > 0\}.$$

For each $(v, \hat{v})$ with $v \prec \hat{v}$, define α by

$$\alpha_{\tilde{v}}^{v\hat{v}} = \begin{cases} \frac{p_v + p_{\hat{v}}}{|\theta_{\hat{v}} - \theta_v|} x^*_{v\hat{v}} & \text{if } \tilde{v} \in \{v, \hat{v}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{48}$$

In other words, the imperfect message $z = v\hat{v}$ is sent only when either v or $\hat{v}$ is realized. To see that α are well-defined probabilities, note that $x^*_{v\hat{v}} > 0$ for at most $K - 1$ pairs $(v, \hat{v})$ with $v \prec \hat{v}$ and $x^*_{v\hat{v}} \leq \frac{\eta}{K-1}$ for any such pair. Hence, for each $v \in V$, the sum of probabilities that imperfect messages is sent at v is given by

$$\begin{aligned} \sum_{\substack{\hat{v} \in V \\ v \prec \hat{v}}} \alpha_v^{v\hat{v}} + \sum_{\substack{\hat{v} \in V \\ v \succ \hat{v}}} \alpha_v^{\hat{v}v} &= \sum_{\substack{\hat{v} \in V \\ v \prec \hat{v} \\ x^*_{v\hat{v}} > 0}} \alpha_v^{v\hat{v}} + \sum_{\substack{\hat{v} \in V \\ \hat{v} \prec v \\ x^*_{\hat{v}v} > 0}} \alpha_v^{\hat{v}v} \\ &= \sum_{\substack{\hat{v} \in V \\ v \prec \hat{v} \\ x^*_{v\hat{v}} > 0}} \frac{p_v + p_{\hat{v}}}{|\theta_{\hat{v}} - \theta_v|} x^*_{v\hat{v}} + \sum_{\substack{\hat{v} \in V \\ \hat{v} \prec v \\ x^*_{\hat{v}v} > 0}} \frac{p_v + p_{\hat{v}}}{|\theta_v - \theta_{\hat{v}}|} x^*_{\hat{v}v} \\ &\leq \frac{1}{\eta} \left\{ \sum_{\substack{\hat{v} \in V \\ v \prec \hat{v}}} x^*_{v\hat{v}} + \sum_{\substack{\hat{v} \in V \\ \hat{v} \prec v}} x^*_{\hat{v}v} \right\} \leq 1. \end{aligned}$$

Finally, α replicates x^* since for the imperfect message $z = v\hat{v}$,

$$\begin{aligned} |\theta_{\hat{v}} - \theta_v| \frac{\alpha_v^{v\hat{v}} \alpha_{\hat{v}}^{v\hat{v}}}{\sum_{\hat{v}} \alpha_{\hat{v}}^{v\hat{v}}} &= |\theta_{\hat{v}} - \theta_v| \frac{\alpha_v^{v\hat{v}} \alpha_{\hat{v}}^{v\hat{v}}}{p_v \alpha_v^{v\hat{v}} + p_{\hat{v}} \alpha_{\hat{v}}^{v\hat{v}}} \\ &= \frac{|\theta_{\hat{v}} - \theta_v| \, \alpha_v^{v\hat{v}}}{p_v + p_{\hat{v}}} = x^*_{v\hat{v}} \quad \text{for any } (v, \hat{v}) \text{ with } v \prec \hat{v}. \end{aligned}$$

We finally show that

$$\{v:\ f(z^r \mid v) > 0\} \neq \{v_{12}, v_{K2}\} \text{ and } \{v_{K1}, v_{K2}\} \text{ for any } r = 1, \ldots, R.$$

Note that these are equivalent to $x^*_{12,K2} = x^*_{K1,K2} = 0$. First, $x_{12,K2} = x_{12,s_K\omega_2}$ (or $x_{K2,12} = x_{s_K\omega_2,12}$) appears when $k = 1$ and when $k = K-1$ in (45), and appears once in each of them: When $k = 1$ and $v = s_K\omega_2$, the coefficient of $x_{v,s_1\omega_2}$ is $-p_v(-1)^{\mathbf{1}_{\{v\prec 12\}}} = -p_v$, and when $k = K-1$ and $v = s_1\omega_2$, the coefficient of $x_{v,s_K\omega_2}$ is $p_v(-1)^{\mathbf{1}_{\{v\prec K2\}}} = -p_v$. Since both coefficients are negative, and the optimal solution must have $x^*_{12,K2} = 0$. Next, $x_{K1,K2} = x_{s_K\omega_1,s_K\omega_2}$ (or $x_{K2,K1} = x_{s_K\omega_2,s_K\omega_1}$) appears only when $k = K-1$ in (45), and appears twice in it: When $v = s_K\omega_1$, the coefficient of $x_{v,s_K\omega_2}$ is $p_v(-1)^{\mathbf{1}_{\{v\prec K2\}}} = -p_v$, and when $v = s_K\omega_2$, the coefficient of $x_{v,s_K\omega_1}$ is $-p_v(-1)^{\mathbf{1}_{\{v\prec K1\}}} = -p_v$. Again, both coefficients are negative and the optimal solution must have $x^*_{K1,K2} = 0$. ■

Proof of Proposition 9.

Using (40), we can express the conditions (4) for cyclical monotonicity as

$$\sum_{i=1}^{n} \left\{ \sum_{\omega} g_{s_{k_i}}(\omega) \left[\theta_{s_{k_i}\omega} + \sum_{v} p_v x_{v,s_{k_i}\omega} (-1)^{\mathbf{1}_{\{v \prec s_{k_i}\omega\}}} \right] \right.$$
$$\left. - \sum_{\omega} g_{s_{k_i}}(\omega) \left[\theta_{s_{k_{i+1}}\omega} + \sum_{v} p_v x_{v,s_{k_{i+1}}\omega} (-1)^{\mathbf{1}_{\{v \prec s_{k_{i+1}}\omega\}}} \right] \right\} \geq 0,$$

which can be rewritten as

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{\omega} g_{s_{k_i}}(\omega) \left[\sum_{v} p_v x_{v,s_{k_i}\omega} (-1)^{\mathbf{1}_{\{v \prec s_{k_i}\omega\}}} - \sum_{v} p_v x_{v,s_{k_{i+1}}\omega} (-1)^{\mathbf{1}_{\{v \prec s_{k_{i+1}}\omega\}}} \right] \\ &\geq \sum_{i=1}^{n} \sum_{\omega} g_{s_{k_i}}(\omega) \left(\theta_{s_{k_{i+1}}\omega} - \theta_{s_{k_i}\omega} \right) \quad \text{for } (k_1, \ldots, k_n) \text{ and } n = 2, \ldots, K. \end{aligned} \tag{49}$$

As noted in (5), (49) is a collection of $N = \sum_{n=2}^{K} \binom{K}{n} (n-1)!$ inequalities which are linear in $x = (x_{v\hat{v}})_{v\prec\hat{v}}$. Since x is itself $\binom{KL}{2}$-dimensional, we can express (49) in matrix form as $Ax \geq \Delta$, where Δ is an N-dimensional vector whose entries are indexed by $(k_1, \ldots, k_n)$ for $n = 2, \ldots, K$ and given by the right-hand side of (49):

$$\Delta_{(k_1,\ldots,k_n)} = \sum_{i=1}^{n} \sum_{\omega} g_{s_{k_i}}(\omega) \left(\theta_{s_{k_{i+1}}\omega} - \theta_{s_{k_i}\omega} \right).$$

For any $h > 0$, when the quality function θ is ε-linear, we see that $\Delta_{(k_1,\ldots,k_n)} \to 0$ as $\varepsilon \to 0$ since by the definition of $(k_1, \ldots, k_n)$, $\sum_{i=1}^{n} \left(k_{i+1} - k_i \right) = 0$. Note also that the matrix A is again a function only of p and $\succcurlyeq$, and independent of the particular choice of $\theta \in \Theta_{\succcurlyeq,\eta}$. The optimization problem with respect to x (corresponding to (29) in the case of 2×2 model) is now given by

$$\min \sum_{\{(v,\hat{v}):\, v \prec \hat{v}\}} x_{v\hat{v}}\, p_v p_{\hat{v}} \, |\theta_{\hat{v}} - \theta_v| \quad \text{subject to: } Ax \geq \Delta,\, x \geq 0. \tag{50}$$

This problem is identical to the problem (46) in the proof of Proposition 8 for the $K \times 2$ case except for the number of inequalities in the constraint set. It follows that the conclusion of the proposition follows if we repeat the argument in the proof of Proposition 8 once we note that the existence of $\bar{x} \geq 0$ with $A\bar{x} \geq \bar{\Delta}$ is now given by Example 7. ■

Online Appendix

Appendix C Mechanism inducing quality enhancement

For simplicity, we assume that the set of action choices is binary $\{0, 1\}$: $e = 0$ and $e = 1$ are interpreted as making no effort and making effort, respectively. Let q_e denote the probability distributions of the agent's p-type when his action is e. We assume that q_1 stochastically dominates q_0 so that the agent is more likely to have a higher p-type when he chooses $e = 1$ than when he chooses $e = 0$. The cost of action e equals c_e and satisfies $c_1 - c_0 > 0$. We assume that conditional on his p-type s, the agent's l-type ω is independent of his action choice, and given by $g_s(\omega)$. Since the agent's incentive in the reporting stage is solely guided by the distribution of the l-type conditional on his p-type, the conditional independence assumption implies that the incentive compatibility of the disclosure mechanism is independent of the agent's action choice. The following lemma records this observation.

Lemma 11 *The mechanism $\Gamma = (y, Z, f)$ is incentive compatible (i.e., satisfies (2)) when the agent's p-type s is distributed according to $q_1 \in \Delta S$ if and only if it is incentive compatible when his p-type is distributed according to $q_0 \in \Delta S$.*

Proof of Lemma 11. Given the disclosure rule (Z, f), the posterior belief ζ_z over V and the quality function θ are both independent of the distribution q of the p-type s, implying that the mean quality μ_z and the ex post compensation $\phi(v) = \sum_z f(z \mid v)\, \mu_z$ are also independent of q. The assumed independence of the probability $g_s(\omega)$ of the l-type ω from the agent's action choice e then shows that the interim compensation $H(s, t) = \sum_\omega g_s(\omega)\, \phi(t, \omega)$ is also independent of q. Hence, the incentive compatibility conditions (2) of Γ are independent of q as well. ■

Let U_e denote the agent's ex ante utility when he chooses action e and reports his p-type truthfully:

$$U_e = \sum_s \Big\{\sum_\omega g_s(\omega)\, \phi(s, \omega) - y(s)\Big\}\, q_e(s) - c_e.$$

An incentive compatible disclosure mechanism $\Gamma = (y, Z, f)$ is *quality-enhancing* if the agent finds it optimal to choose $e = 1$ conditional on truthful reporting, or equivalently,

$$U_1 \geq U_0. \tag{51}$$

By Lemma 11, if Γ is incentive compatible and quality-enhancing, then the agent has no profitable combinatorial deviation, where he first chooses $e = 0$ and then misreports his p-type.[36] A quality-enhancing incentive compatible mechanism is *optimal* if it minimizes the quadratic loss function (3) among the class of such mechanisms. A disclosure rule (Z, f) is *quality-enhancing* if there exists a cost-assignment rule y for which $\Gamma = (y, Z, f)$ is quality-enhancing, and is optimal if it corresponds to an optimal quality-enhancing incentive compatible mechanism.

For simplicity, the analysis in what follows restricts attention to the 2×2 environment of Section 6.1 where both the p-type and l-type are binary. The four profiles are named as in (8). Stochastic dominance of q_1 over q_0 is equivalent to $q_1(s_2) > q_0(s_2)$, and (51) can be rewritten as

$$y(s_2) - y(s_1) \leq \sum_{\omega} \left\{ \phi(s_2, \omega)\, g_{s_2}(\omega) - \phi(s_1, \omega)\, g_{s_1}(\omega) \right\} - \frac{c_1 - c_0}{q_1(s_2) - q_0(s_2)}. \qquad (52)$$

Clearly, it is not possible to induce the agent to choose $e = 1$ if it is very costly compared with the corresponding increase in the expected compensation. We introduce the following measure to quantify the effect of the cost of $e = 1$:

$$C = \frac{c_1 - c_0}{\{g_{s_2}(\omega_2) - g_{s_1}(\omega_2)\}\{q_1(s_2) - q_0(s_2)\}} - (\theta_4 - \theta_3). \qquad (53)$$

As seen, C measures the difference between the marginal cost of $e = 1$ ($c_1 - c_0$, adjusted by the probabilities) and the maximum differential ($\theta_4 - \theta_3$) in the compensation when the agent has the high p-type s_2. The following proposition describes a sufficient condition in terms of C for the feasibility of an quality-enhancing mechanism. In line with our intuition developed in the previous sections, the number of pooling messages required under the optimal mechanism is related to the number of incentive conditions faced by the agent. Since (51) places one additional constraint compared with the baseline 2×2 model, it follows that the optimal mechanism entails at most two pooling messages as shown in the following proposition.

Proposition 12 *Suppose that the quality function θ is submodular ($\Delta > 0$) and that (θ, p) satisfies either one of (25), (26), and (27) of Proposition 6. Furthermore, suppose that C satisfies*

$$\begin{array}{lll} C \leq \frac{p_1(\theta_3 - \theta_1)}{p_1 + p_3}, & & \\ C \leq \frac{p_2|\theta_3 - \theta_2|}{p_2 + p_3} & \textit{if } \theta_2 \neq \theta_3, & \\ C \leq \frac{p_2(\theta_4 - \theta_3)(\theta_3 - \theta_2)}{(p_3 - p_2)(\theta_3 - \theta_2) + (p_2 + p_3)(\theta_2 - \theta_1)} & \textit{if } \theta_2 < \theta_3 \textit{ and } p_2 < p_3. & \end{array} \qquad (54)$$

[36] By (51), the agent cannot profitably deviate by choosing $e = 0$ and then truthfully reporting his p-type. By Lemma 11, (2) ensures that the agent cannot profitably misreport his p-type regardless of his action choice e.

Then there exists an optimal quality-enhancing disclosure rule (Z, f) *with at most two pooling messages:*

$$Z = V \cup \{z_1, z_2\} \quad \textit{for } z_1,\ z_2 \notin V.$$

The support of each pooling message is binary and the combination of the support of z_1 *and* z_2 *is given by*

$$\big(\mathrm{supp}(z_1),\, \mathrm{supp}(z_2)\big) \in \big\{(\{1,2\},\{1,3\}),\ (\{1,2\},\{2,3\}),\ (\{1,3\},\{2,3\})\big\}.$$

Proof of Proposition 12. By Lemma 11, incentive compatibility of a disclosure mechanism is independent of the agent's action choice e. When the p-type is binary, the mechanism $\Gamma = (y, Z, f)$ is incentive compatible if and only if

$$\begin{aligned}\sum_{\omega} \big\{\phi(s_2,\omega) - \phi(s_1,\omega)\big\}\, g_{s_1}(\omega) &\le y(s_2) - y(s_1) \\ &\le \sum_{\omega} \big\{\phi(s_2,\omega) - \phi(s_1,\omega)\big\}\, g_{s_2}(\omega).\end{aligned} \tag{55}$$

It follows that an incentive compatible quality-enhancing mechanism $\Gamma = (y, Z, f)$ exists if and only if (52) and (55) hold, or equivalently, ϕ is supermodular, and

$$\begin{aligned}&\sum_{\omega} \big\{\phi(s_2,\omega) - \phi(s_1,\omega)\big\}\, g_{s_1}(\omega) \\ &\le \sum_{\omega} \big\{\phi(s_2,\omega)\, g_{s_2}(\omega) - \phi(s_1,\omega)\, g_{s_1}(\omega)\big\} - \frac{c_1 - c_0}{q_1(s_2) - q_0(s_2)}.\end{aligned} \tag{56}$$

Simplifying (56) using Lemma 5 as well as the fact that the l-type ω is binary, we see that the disclosure rule (Z, f) is quality-enhancing if and only if

$$\begin{aligned}&x_{12}\,(p_1 + p_2) + x_{13}\,(p_1 + p_3) - x_{24}\,(p_2 + p_4) - x_{34}\,(p_3 + p_4) \\ &+ x_{14}\,(p_4 - p_1) - (-1)^{\mathbf{1}_{\{2 \prec 3\}}} x_{23}\,(p_2 - p_3) \ge \Delta.\end{aligned} \tag{57}$$

and

$$-p_1 x_{14} - p_2 x_{24} - (p_3 + p_4) x_{34} + p_1 x_{13} - (-1)^{\mathbf{1}_{\{2 \prec 3\}}}\, p_2 x_{23} \ge C, \tag{58}$$

where C is as defined in (53).[37]

Let x^* denote the optimal solution. Clearly, $x^*_{24} = x^*_{34} = 0$ since their coefficients are negative in both (57) and (58). It follows that only x^*_{12}, x^*_{13}, x^*_{14}, and x^*_{23} can be strictly positive, and by the same logic as in the proof of Proposition 8, we can take x^* so that at most two of them are strictly positive.

[37] Note that (57) is identical to (43) in the proof of Proposition 6.

We begin by showing that $x^*_{14} = 0$. Suppose to the contrary that $p_4 - p_1 > 0$ and $x^*_{14} > 0$. For (58) to hold, it must be the case that either $x^*_{13} > 0$ or $x^*_{23} > 0$. If $x^*_{14} > 0$, then, $x^*_{v\hat{v}} > 0$ for two coordinates $(v, \hat{v})$. We can then assume that x^* satisfies both (57) and (58) with equality since otherwise, there would exist a solution x^* such that $x^*_{v\hat{v}} > 0$ only for a single coordinate $(v, \hat{v})$.

- $x^*_{13} > 0$, $x^*_{14} > 0$, and $x^*_{v\hat{v}} = 0$ for $(v, \hat{v}) \neq (1, 3)$, $(1, 4)$.

  Let $\hat{x}$ and $\tilde{x}$ be such that

  $$\begin{aligned} \hat{x}_{13} &= \tfrac{\Delta}{p_1+p_3}, \text{ and } \hat{x}_{v\hat{v}} = 0 \text{ for } (v, \hat{v}) \neq (1, 3), \\ \tilde{x}_{14} &= \tfrac{\Delta}{p_4-p_1}, \text{ and } \tilde{x}_{v\hat{v}} = 0 \text{ for } (v, \hat{v}) \neq (1, 4). \end{aligned}$$

  Note that both $\hat{x}$ and $\tilde{x}$ satisfy (57) with equality. Since x^* also satisfies (57) with equality, x^* is a linear combination of $\hat{x}$ and $\tilde{x}$. Furthermore, $(p_4 - p_1)\, x^*_{14} = \Delta - (p_1 + p_3)\, x^*_{13} > 0$ so that $x^*_{13} < \frac{\Delta}{p_1+p_3}$, which implies that

  $$p_1\hat{x}_{13} - p_1\hat{x}_{14} = p_1 \frac{\Delta}{p_1 + p_3} > p_1 x^*_{13} - p_1 x^*_{14} = C.$$

  In other words, $\hat{x}$ satisfies (58). The inequalities $\frac{p_3}{p_1+p_3} < \frac{p_4}{p_4-p_1}$ and $\theta_3 < \theta_4$ together imply

  $$\begin{aligned} p_1 p_3\, (\theta_3 - \theta_1)\, \hat{x}_{13} &= p_1 p_3\, (\theta_3 - \theta_1) \frac{\Delta}{p_1 + p_3} \\ &< p_1 p_4\, (\theta_4 - \theta_1) \frac{\Delta}{p_4 - p_1} \\ &= p_1 p_4\, (\theta_4 - \theta_1)\, \tilde{x}_{14}, \end{aligned}$$

  and hence $\mathcal{L}(\hat{x}) < \mathcal{L}(\tilde{x})$. Since x^* is a convex combination of $\hat{x}$ and $\tilde{x}$ as noted above and since $\mathcal{L}$ is linear, $\mathcal{L}(x^*)$ is also a convex combination of $\mathcal{L}(\hat{x})$ and $\mathcal{L}(\tilde{x})$, and satisfies $\mathcal{L}(x^*) > \mathcal{L}(\hat{x})$. Given that $\hat{x}$ satisfies both (57) and (58), this is a contradiction to the optimality of x^*.

- $x^*_{14} > 0$, $x^*_{23} > 0$, and $x^*_{v\hat{v}} = 0$ for $(v, \hat{v}) \neq (2, 3)$, $(1, 4)$.

  Let $\hat{x}$ be such that $\hat{x}_{13} = \frac{p_4-p_1}{p_1+p_3}\, x^*_{14}$, $\hat{x}_{14} = 0$, and $\hat{x}_{v\hat{v}} = x^*_{v\hat{v}}$ for $(v, \hat{v}) \neq (1, 3)$, $(1, 4)$. $\hat{x}$ then satisfies both (57) and (58). Since $\frac{p_3}{p_1+p_3} < \frac{p_4}{p_4-p_1}$ and $\theta_3 < \theta_4$, we have

  $$p_1 p_3 (\theta_3 - \theta_1)\, \hat{x}_{13} = p_1 p_3 (\theta_3 - \theta_1) \frac{p_4 - p_1}{p_1 + p_3}\, x^*_{14} < p_1 p_4 (\theta_4 - \theta_1)\, x^*_{14},$$

  which leads to the contradiction that $\mathcal{L}(\hat{x}) < \mathcal{L}(x^*)$.

We hence conclude that at most two of x_{12}^*, x_{13}^*, and x_{23}^* can be strictly positive, and proceed by examining the following three cases separately.

1) If $x_{12}^* = 0$, then (57) and (58) reduce to

$$x_{13}^* (p_1 + p_3) - (-1)^{\mathbf{1}_{\{2 \prec 3\}}} x_{23}^* (p_2 - p_3) \geq \Delta, \tag{59}$$

$$p_1 x_{13}^* - (-1)^{\mathbf{1}_{\{2 \prec 3\}}} p_2 x_{23}^* \geq C. \tag{60}$$

At least one of these two inequalities with equality.

(a) Suppose first that (60) holds with equality.

- If $2 \prec 3$ ($\Leftrightarrow \theta_2 < \theta_3$), then (60) becomes

$$p_1 x_{13}^* + p_2 x_{23}^* = C.$$

By (54), both $(x_{13}, x_{23}) = (\frac{C}{p_1}, 0)$ and $(x_{13}, x_{23}) = (0, \frac{C}{p_2})$ satisfy

$$\frac{p_1 + p_3}{\theta_3 - \theta_1} x_{13} + \frac{p_2 + p_3}{|\theta_3 - \theta_2|} x_{23} \leq 1. \tag{61}$$

(x_{13}^*, x_{23}^*) is a convex combination of these two points, and hence satisfies (61) as well.

- If $2 \succcurlyeq 3$ ($\Leftrightarrow \theta_2 \geq \theta_3$), then (60) becomes

$$p_1 x_{13}^* - p_2 x_{23}^* = C.$$

If $p_2 \geq p_3$ so that (25) holds, then $x_{23}^* = 0$ and $x_{13}^* = \max\{\frac{C}{p_1}, \frac{\Delta}{p_1+p_3}\}$. By (54), x^* satisfies (61). Assume then in what follows that (25) does not hold so that $\theta_2 > \theta_3$ and $p_2 < p_3$. If x^* satisfies (59) with strict inequality, then $(x_{13}^*, x_{23}^*) = (\frac{C}{p_1}, 0)$, which satisfies (61) by (54). On the other hand, suppose x^* satisfies (59) also with inequality. If (26) holds, then $(x_{13}^*, x_{23}^*) = (\frac{\Delta}{p_1+p_3}, 0)$, which satisfies (61). If (26) does not hold, then (27) holds by assumption. Note that x^* in this case is a convex combination of $(x_{13}, x_{23}) = (\frac{\Delta}{p_1+p_3}, 0)$ and $(x_{13}, x_{23}) = (0, \frac{\Delta}{p_3-p_2})$, both of which satisfy (61) under (27). It follows that x^* also satisfies (61).

(b) If (60) holds with strict inequality, then x^* is identical to the optimal solution in the proof of Proposition 6, and satisfies (61) under (25), (26) or (27).

Take the number of pooling messages $R = 2$ and let the probability $\alpha_v^r = f(z^r \mid v)$ of message z^r given profile $v \in V$ ($r = 1, 2$ and $v = 1, 2, 3$) be defined by

$$\alpha_2^1 = 0, \quad \alpha_1^1 = \alpha_3^1 = \frac{p_1 + p_3}{\theta_3 - \theta_1} x_{13}^*, \quad \text{and} \quad \alpha_1^2 = 0, \quad \alpha_2^2 = \alpha_3^2 = \frac{p_2 + p_3}{\theta_3 - \theta_2} x_{23}^*.$$

Then α_v^r is well-defined since $\alpha_3^1 + \alpha_3^2 \leq 1$ by (61), and hence also $\alpha_1^1 + \alpha_1^2$, $\alpha_2^1 + \alpha_2^2 \leq 1$. Furthermore, it replicates x^* since

$$\sum_{r=1}^{2} \frac{\alpha_1^r \alpha_3^r}{p_1 \alpha_1^r + p_3 \alpha_3^r} = \frac{\alpha_1^1}{p_1 + p_3} = x_{13}^*, \quad \sum_{r=1}^{2} \frac{\alpha_2^r \alpha_3^r}{p_2 \alpha_2^r + p_3 \alpha_3^r} = \frac{\alpha_2^2}{p_2 + p_3} = x_{23}^*.$$

2) If $x_{13}^* = 0$, then (57) and (58) reduce to

$$x_{12}^* (p_1 + p_2) - (-1)^{\mathbf{1}_{\{2 \prec 3\}}} x_{23}^* (p_2 - p_3) \geq \Delta, \tag{62}$$

$$-(-1)^{\mathbf{1}_{\{2 \prec 3\}}} p_2 x_{23}^* \geq C. \tag{63}$$

Again, at least one of these two inequalities with equality. Note that (63) requires that $2 \prec 3$ ($\Leftrightarrow \theta_2 < \theta_3$).

(a) Suppose first that (63) holds with equality:

$$p_2 x_{23}^* = C.$$

If (62) holds with strict inequality, then $(x_{12}^*, x_{23}^*) = (0, \frac{C}{p_2})$. By (54), x^* satisfies

$$\frac{p_1 + p_2}{\theta_2 - \theta_1} x_{12}^* + \frac{p_2 + p_3}{|\theta_3 - \theta_2|} x_{23}^* \leq 1. \tag{64}$$

Suppose now that (62) also holds with equality.

- $p_2 > p_3$. In this case, (25) does not hold. If (26) holds, then $(x_{12}^*, x_{23}^*) = (\frac{\Delta}{p_1 + p_2}, 0)$. If (26) does not hold either, then (27) holds by assumption and (x_{12}^*, x_{23}^*) is a convex combination of $(x_{12}, x_{23}) = (0, \frac{\Delta}{p_2 - p_3})$ and $(x_{12}, x_{23}) = (\frac{\Delta}{p_1 + p_2}, 0)$, and satisfies (61) since both these points satisfy (64) under (27).
- $p_2 \leq p_3$. In this case,

$$(x_{12}^*, x_{23}^*) = \Big(\frac{1}{p_1 + p_2} \Big\{ \Delta + \frac{p_3 - p_2}{p_2} C \Big\}, \frac{C}{p_2} \Big).$$

  x^* then satisfies (64) because of the third condition in (54).

(b) If (63) holds with strict inequality, then x^* is identical to the optimal solution in the proof of Proposition 6 and satisfies (64) under (25), (26) or (27).

If we let $R = 2$ and define $\alpha_v^r = f(z^r \mid v)$ $(r = 1, 2)$ by

$$\alpha_3^1 = 0, \quad \alpha_1^1 = \alpha_2^1 = \frac{p_1 + p_2}{\theta_2 - \theta_1} x_{12}^*, \quad \text{and} \quad \alpha_1^2 = 0, \quad \alpha_2^2 = \alpha_3^2 = \frac{p_2 + p_3}{|\theta_3 - \theta_2|} x_{23}^*,$$

then α_v^r is a well-defined probability by (64) and replicates x^* as above.

3) If $x_{23}^* = 0$, then (57) and (58) reduce to

$$x_{12}^* (p_1 + p_2) + x_{13}^* (p_1 + p_3) \geq \Delta, \tag{65}$$
$$p_1 x_{13}^* \geq C. \tag{66}$$

Again, the optimal $x = x^*$ satisfies at least one of these two inequalities with equality.

(a) If (66) holds with equality and (65) holds with strict inequality, then $(x_{12}^*, x_{13}^*) = (0, \frac{C}{p_1})$. By (54), x^* satisfies

$$\frac{p_1 + p_2}{\theta_2 - \theta_1} x_{12}^* + \frac{p_1 + p_3}{\theta_3 - \theta_1} x_{13}^* \leq 1. \tag{67}$$

(b) If x^* satisfies both (66) and (65) with equality, then it satisfies (67) because it is a convex combination of $(x_{12}, x_{13}) = (\frac{\Delta}{p_1+p_2}, 0)$ and $(x_{12}, x_{13}) = (0, \frac{\Delta}{p_1+p_3})$, both of which satisfy (67).

(c) If x^* satisfies (66) with strict inequality, then x^* is identical to the optimal solution in the proof of Proposition 6 and satisfies (67) under (25), (26) or (27).

If we let $R = 2$ and define $\alpha_v^r = f(z^r \mid v)$ $(r = 1, 2)$ by

$$\alpha_3^1 = 0, \quad \alpha_1^1 = \alpha_2^1 = \frac{p_1 + p_2}{\theta_2 - \theta_1} x_{12}^*, \quad \text{and} \quad \alpha_2^2 = 0, \quad \alpha_1^2 = \alpha_3^2 = \frac{p_1 + p_3}{\theta_3 - \theta_1} x_{13}^*,$$

then α_v^r is a well-defined probability by (67) and replicates x^* as above.

In all the three cases above, hence, we can take $R = 2$ and α_v^r to be positive for at most two profiles for each r. This completes the proof. ■

Appendix D Maximizing the probability of compensation under IC

Assume that the agent receives a fixed compensation $W = 1$ if and only if his expected quality is at or above $\underline{\theta}$, where $\underline{\theta} > E_v[\theta_v]$. Let $\Pi(\Gamma)$ denote the probability that the agent gets receives the compensation under the mechanism Γ:

$$\Pi(\Gamma) = E_{v,z}[\mathbf{1}_{\{\mu_z \geq \underline{\theta}\}}] = \sum_v p_v \sum_z f(z \mid v) \, \mathbf{1}_{\{\mu_z \geq \underline{\theta}\}}.$$

The optimal mechanism Γ^* maximizes the probability of compensation under the agent's incentive compatibility conditions:

$$\Gamma^* \in \underset{\Gamma}{\operatorname{argmax}} \left\{ \Pi(\Gamma) : \Gamma \text{ satisfies (IC)} \right\}.$$

By a version of the revelation principle, it is without loss of generality to restrict attention to disclosure rules (Z, f) with the binary message space $Z = \{z_0, z_1\}$: z_1 is the recommendation for compensation, and z_0 is the recommendation for no compensation:

$$\mu_{z_0} < \underline{\theta}, \quad \text{and} \quad \mu_{z_1} \geq \underline{\theta}. \tag{68}$$

Let $\tau_f(\psi_{v,\zeta}) = \hat{\tau}(\zeta)\,\zeta(v)$ be the consistent dual-belief distribution corresponding to the disclosure rule (Z, f) with $Z = \{z_0, z_1\}$ (Appendix A). Then by Proposition 10, we can express the principal's objective function as

$$\Pi(\Gamma) = \sum_v p_v f(z_1 \mid v) = \sum_v \hat{\tau}(\zeta_{z_1})\,\zeta_{z_1}(v) = \hat{\tau}(\zeta_{z_1}),$$

and the ex post compensation function ϕ as

$$\phi(v) = f(z_1 \mid v) = \frac{1}{p_v}\,\hat{\tau}(\zeta_{z_1})\,\zeta_{z_1}(v). \tag{69}$$

Furthermore, the expected quality given messages z_0 and z_1 equals

$$\mu_{z_0} = \sum_v \zeta_{z_0}(v)\theta_v, \quad \text{and} \quad \mu_{z_1} = \sum_v \zeta_{z_1}(v)\theta_v.$$

In what follows, consider for simplicity the 2×2 environment of Section 6.1 and name the profiles as in (8). It then readily follows from Lemma 3 and (69) that the disclosure rule (Z, f) is implementable if and only if

$$\phi(v_2) - \phi(v_1) \leq \phi(v_4) - \phi(v_3) \quad \Leftrightarrow \quad \frac{y_2}{p_2} - \frac{y_1}{p_1} \leq \frac{y_4}{p_4} - \frac{y_3}{p_3}, \tag{70}$$

where $y_m = \zeta_{z_1}(v_m)$ $(m = 1, \ldots, 4)$ is the posterior belief weight on profile v following z_1. Hence, implementability in this case translates to a direct restriction on the posterior belief after z_1. Summarizing, the principal's problem can be formulated as:

$$\max_{\hat{\tau}(\zeta_{z_1}), y, \zeta_{z_0}} \quad \hat{\tau}(\zeta_{z_1})$$

$$\text{subject to:} \begin{cases} \frac{y_2}{p_2} - \frac{y_1}{p_1} \leq \frac{y_4}{p_4} - \frac{y_3}{p_3}, & \text{(implementability)} \\ \sum_m y_m \theta_m \geq \underline{\theta}, & \text{(obedience)} \\ (1 - \hat{\tau}(\zeta_{z_1}))\,\zeta_{z_0} + \hat{\tau}(\zeta_{z_1})\,y = p, & \text{(consistency)} \\ y \geq 0, \sum_m y_m = 1, \hat{\tau}(\zeta_{z_1}) \in [0,1]. & \text{(probability requirements)} \end{cases}$$

Assume now for concreteness that

$$\theta_1 < \theta_2 < \underline{\theta} < \theta_3 < \theta_4.$$

Define

$$\alpha_m = f(z_1 \mid v_m) = \hat{\tau}(\zeta_{z_1})\,\frac{y_m}{p_m} \quad \text{for } m = 1, \ldots, 4.$$

Using α_m, we can rewrite $\Pi(\Gamma)$ and (70) respectively as:

$$\Pi(\Gamma) = \sum_m p_m \alpha_m, \quad \text{and} \quad \alpha_4 - \alpha_3 \geq \alpha_2 - \alpha_1.$$

Posterior beliefs y and ζ_{z_0} are determined by α and satisfy both consistency and probability requirements. We can then verify the following:

Lemma 13 *For the optimal disclosure rule (Z, f), the implementability condition (70) is binding, and $\mu_{z_1} = \underline{\theta}$.*

Proof. If $\alpha_4 < 1$, then a small increase in α_4 increases both $\Pi(\Gamma)$ and μ_{z_1} (and decreases μ_{z_0}) without violating (70). Hence, $\alpha_4 = 1$.

Suppose $\alpha_2 - \alpha_1 < \alpha_4 - \alpha_3$. If $\alpha_3 < 1$, then a small increase in α_3 increases both $\Pi(\Gamma)$ and μ_{z_1} (and decreases μ_{z_0}) while maintaining (70). If $\alpha_3 = \alpha_4 = 1$, then $\alpha_2 - \alpha_1 < 0$. Let $\alpha'_3 = \alpha'_4 = 1$, $\alpha'_2 = \alpha_2 + \varepsilon_2$ and $\alpha'_1 = \alpha_1 - \varepsilon_1$ for ε_1, $\varepsilon_2 > 0$ small. If $p_2\varepsilon_2 > p_1\varepsilon_1$, this increases $\Pi(\Gamma)$ and μ_{z_1}.[38] We hence conclude that (70) holds with equality.

[38]Let μ'_{z_1} denote the conditional mean corresponding to α'. Since $\alpha'_1 < \alpha_1$ and $\alpha'_2 > \alpha_2$,

$$\frac{\alpha_1 p_1 \theta_1 + \alpha_2 p_2 \theta_2}{\alpha_1 p_1 + \alpha_2 p_2} < \frac{\alpha'_1 p_1 \theta_1 + \alpha'_2 p_2 \theta_2}{\alpha'_1 p_1 + \alpha'_2 p_2}.$$

This along with $\alpha_1 p_1 + \alpha_2 p_2 < \alpha'_1 p_1 + \alpha'_2 p_2$ implies

$$\begin{aligned}
\mu_{z_1} &= \frac{\sum_m \alpha_m p_m \theta_m}{\sum_m \alpha_m p_m} \\
&= \frac{\frac{\alpha_1 p_1 \theta_1 + \alpha_2 p_2 \theta_2}{\alpha_1 p_1 + \alpha_2 p_2}(\alpha_1 p_1 + \alpha_2 p_2) + p_3\theta_3 + p_4\theta_4}{(\alpha_1 p_1 + \alpha_2 p_2) + p_3\theta_3 + p_4\theta_4} \\
&< \frac{\frac{\alpha'_1 p_1 \theta_1 + \alpha'_2 p_2 \theta_2}{\alpha'_1 p_1 + \alpha'_2 p_2}(\alpha_1 p_1 + \alpha_2 p_2) + p_3\theta_3 + p_4\theta_4}{(\alpha_1 p_1 + \alpha_2 p_2) + p_3\theta_3 + p_4\theta_4} \\
&< \frac{\frac{\alpha'_1 p_1 \theta_1 + \alpha'_2 p_2 \theta_2}{\alpha'_1 p_1 + \alpha'_2 p_2}(\alpha'_1 p_1 + \alpha'_2 p_2) + p_3\theta_3 + p_4\theta_4}{(\alpha'_1 p_1 + \alpha'_2 p_2) + p_3\theta_3 + p_4\theta_4} \\
&= \frac{\sum_m \alpha'_m p_m \theta_m}{\sum_m \alpha'_m p_m} = \mu'_{z_1}.
\end{aligned}$$

Suppose next that $\mu_{z_1} > \underline{\theta}$. Suppose $\alpha_1 = \alpha_2 = 1$. Then $\alpha_3 = \alpha_4$ since (70) holds with equality. We then have a violation of $\mu_{z_1} \geq \underline{\theta}$ since

$$\mu_{z_1} = \frac{p_1\theta_1 + p_2\theta_2 + \alpha_3(p_3\theta_3 + p_4\theta_4)}{p_1 + p_2 + \alpha_3(p_3 + p_4)} \leq \frac{\sum_m p_m\theta_m}{\sum_m p_m} = E_v[\theta_v] < \underline{\theta}.$$

where the last inequality holds by assumption. If $\alpha_1 = \alpha_2 < 1$, then a small increase in both α_2 and α_1 by the same amount increases $\Pi(\Gamma)$ while maintaining (70) and (68). If $\alpha_1 < \alpha_2$, then a small increase in α_1 increases $\Pi(\Gamma)$ while maintaining (70) and (68). We hence conclude that $\mu_{z_1} = \underline{\theta}$. ■

By Lemma 13, $\alpha_1 = \alpha_2 + \alpha_3 - 1$, and

$$\mu_{z_1} = \frac{(p_1\theta_1 + p_2\theta_2)\,\alpha_2 + (p_1\theta_1 + p_3\theta_3)\,\alpha_3 + (p_4\theta_4 - p_1\theta_1)}{(p_1 + p_2)\,\alpha_2 + (p_1 + p_3)\,\alpha_3 + (p_4 - p_1)} = \underline{\theta}. \tag{71}$$

The principal's problem is then written as:

$$\max_{\alpha_2,\alpha_3} (p_1 + p_2)\,\alpha_2 + (p_1 + p_3)\,\alpha_3 + (p_4 - p_1) \tag{72}$$

$$\text{subject to: (71) and } \alpha_2 + \alpha_3 \geq 1. \tag{73}$$

Define

$$\gamma_2 = \frac{p_4(\theta_4 - \underline{\theta}) - p_1(\theta_1 - \underline{\theta})}{-p_2(\theta_2 - \underline{\theta}) - p_1(\theta_1 - \underline{\theta})},$$
$$\gamma_3 = \frac{p_4(\theta_4 - \underline{\theta}) - p_1(\theta_1 - \underline{\theta})}{-p_3(\theta_3 - \underline{\theta}) - p_1(\theta_1 - \underline{\theta})},$$
$$\gamma_4 = \frac{p_3(\theta_3 - \underline{\theta}) + p_4(\theta_4 - \underline{\theta})}{p_3(\theta_3 - \underline{\theta}) - p_2(\theta_2 - \underline{\theta})}.$$

In the (α_2, α_3)-plane, $(\gamma_2, 1)$ is the point of intersection between (71) and $\alpha_3 = 1$, $(1, \gamma_3)$ is the point of intersection between (71) and $\alpha_2 = 1$, and $(\gamma_4, 1 - \gamma_4)$ is the point of intersection between (71) and $\alpha_2 + \alpha_3 = 1$. Noting that all the functions appearing in (73) are linear in (α_2, α_3), we conclude that there exists an optimal disclosure rule with α_v corresponding to these points of intersection. In other words, there exists an optimal disclosure rule for which $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is given by[39]

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \{(\gamma_2, \gamma_2, 1, 1), (\gamma_3, 1, \gamma_3, 1), (0, \gamma_4, 1 - \gamma_4, 1)\}.$$

This is illustrated in Figure 7.

Without the implementability condition (70), the optimal disclosure rule would set $\alpha_3 = \alpha_4 = 1$ and $\alpha_1 < \alpha_2$ since $\theta_1 < \theta_2$. For example, if $E_v[\theta_v \mid v \neq 1] < \underline{\theta}$, then we would have $\alpha_1 = 0 < \alpha_2 < 1$. It follows that (70) entails a loss in the principal's payoff compared with when the agent p-type is directly observed.

[39] $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\gamma_2, \gamma_2, 1, 1)$ is feasible (i.e., $\gamma_2 \in (0, 1]$) under our assumption that $\underline{\theta} < \theta_3 < \theta_4$. Feasibility of other cases depends on parameters.

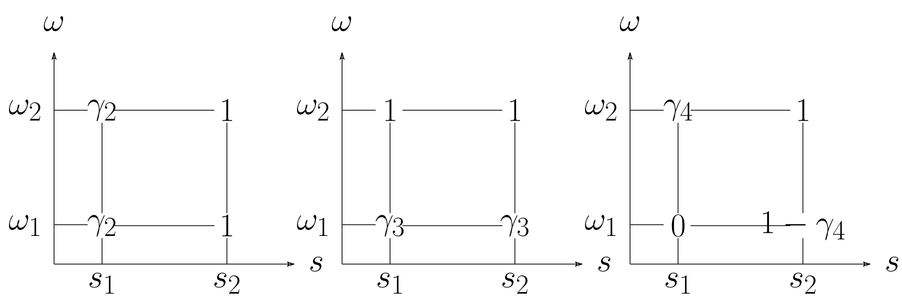

Figure 7: Optimal disclosure rules under the alternative objective function

The numbers at each profile v indicate the probability $\alpha_v = f(z_1 \mid v)$.

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