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Procedural and Optimization Implementation of the Weighted ENSC Value

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Abstract

The main purpose of this article is to introduce the weighted ENSC value for cooperative transferable utility games which takes into account players' selfishness about the payoff allocations. Similarly to Shapley's idea of a one-by-one formation of the grand coalition (Shapley, 1953 [13]), we first provide a procedural implementation of the weighted ENSC value depending on players' selfishness as well as their marginal contributions to the grand coalition. Second, in the spirit of the nucleolus (Schmeidler, 1969 [12]), we prove that the weighted ENSC value is obtained by lexicographically minimizing a complaint vector associated with a new complaint criterion relying on players' selfishness.

Keywords: TU-game; weighted ENSC value; allocation scenario; selfish complaint
JEL Classification System: C71

1 Introduction

In the classical cooperative game theory, it is generally assumed that players are totally selfish when distributing the worth generated by cooperating activities. Undoubtedly, the most prominent example of players' pure selfishness is the Shapley value (Shapley, 1953 [13]). Assuming that the grand coalition has formed by a succession of one-by-one arrivals, the Shapley value is obtained when each entering player obtains his entire expected

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marginal contribution to the coalition of players he joins upon arriving.[1] Yet in many real-life situations, individuals may show solidarity with the others to ensure that each gets his "slice of the pie".[2]

In this article, we provide a new allocation rule, called the weighted Equal Allocation of Non-Separable Contributions (ENSC) value and inspired by the ENSC value[3] (Moulin, 1985 [9]), which takes into account players' selfishness when distributing the worth of the grand coalition. Interestingly, the weighted ENSC value respectively coincides with the ENSC value when players are totally selfish and the equal division solution when players are purely altruistic.

First, we implement the weighted ENSC value by designing a new allocation scenario based on a one-by-one formation of the grand coalition in which each entering player takes away a part of his marginal contribution to the grand coalition depending on his selfishness level, while the residual brought by his joining is evenly distributed among the players already there. Our construction procedure is justified by the necessity to appeal to a regulator to ensure some solidarity among the players as is the case, for example, with health insurances (Stone, 1993 [14]).

Once the players have received their payoffs in a transferable utility game (henceforth a TU-game), they may be dissatisfied and express an amount of complaint within a coalition.[4] The complaint of a coalition manifests its members' potential to create more utility if they decide to contest an allocation. When the sum of players' marginal contributions to the grand coalition cannot be covered by the worth of the grand coalition, we prove that the weighted ENSC value is obtained by lexicographically minimizing a complaint vector associated with a new complaint criterion, called the selfish complaint, which only depends on players' selfishness as well as their marginal contributions to the grand coalition.

The rest of the article is organized as follows. Section 2 provides basic definitions and notations. Section 3 deals with the procedural implementation of the weighted ENSC value. In Section 4, we study the weighted ENSC value by using an optimization approach based on players' complaint. Finally, Section 5 concludes.

2 TU-games and solution concepts

Let $N = \{1, 2, \ldots, n\}$ be a fixed and finite set of n players. An element $i \in N$ and a non-empty subset $S \subseteq N$ are respectively called a player and a coalition. For each coalition $S \subseteq N$, its cardinality will be denoted by $|S|$. A **TU-game** on the set N of players is a real-valued function v defined on the set 2^N of all subsets of N such that $v(\emptyset) = 0$. For

[1] Malawski's procedural values (Malawski, 2013 [8]) and Sun *et al.*'s procedure interpretation of the ENSC value (Sun *et al.*, 2017 [15]) also consider that each entrant captures his entire marginal contribution.

[2] Among others, the results of Guth *et al.* (1982 [7]), Forsythe *et al.* (1994 [6]), Nowak *et al.* (2000 [10]), and Chen *et al.* (2014 [2]) reinforce this view.

[3] Recently, an interesting characterization of the ENSC value was proposed by Béal *et al.* (2016 [1]) using the axiom of balanced collective contributions together with the classical axiom of efficiency.

[4] Davis and Maschler (1965 [3]) introduced the complaint of a coalition based on the excess criterion to determine the kernel of TU-games.

each coalition $S \subseteq N$, $v(S)$ describes the worth that coalition S can achieve when all its members cooperate. The set of all TU-games is denoted by G.

An **order** on N is a bijection $\pi : N \longrightarrow N$ where $\pi(l) = i$ indicates that player i has the l^{th} position. The set of all the $n!$ orders is denoted by $\Pi(N)$.

A **solution** on G is a function φ which associates with each TU-game $(N, v) \in G$ a subset $\varphi(N, v) \subseteq R^N$ of payoff vectors. If φ assigns a unique payoff vector to each TU-game $(N, v) \in G$, then φ is called a **value**.

For a TU-game $(N, v) \in G$, the **pre-imputation set** $I^*(N, v)$ (Driessen, 1988 [5]) is defined as the set of payoff vectors that fully distribute the worth of the grand coalition among the players, i.e.:

$$I^*(N, v) = \{x \in R^N \mid x(N) = v(N)\}.$$

The **ENSC value**, introduced by Moulin (1985 [9]), gives to each player $i \in N$ an equal share of the surplus generated by the grand coalition besides his marginal contribution to the grand coalition, i.e.:

$$ENSC_i(N, v) = b_i^v + \frac{1}{n}\Big(v(N) - \sum_{j \in N} b_j^v\Big), \quad \text{for all } i \in N,$$

where $b_i^v = v(N) - v(N \setminus \{i\})$.

The **weighted ENSC value** we propose generalizes the ENSC value and allocates to each player $i \in N$ an equal share of the residual surplus generated by the grand coalition besides a part $\alpha_i \in [0, 1]$ of his marginal contribution to the grand coalition b_i^v, i.e.:

$$ENSC_i^{\alpha}(N, v) = \alpha_i b_i^v + \frac{1}{n}\Big(v(N) - \sum_{j \in N} \alpha_j b_j^v\Big), \quad \text{for all } i \in N.$$

The parameter α_i is interpreted as the level of player i's selfishness and represents the part of his marginal contribution to the grand coalition he would like to obtain. Particularly, if $\alpha_i = 0$ for all $i \in N$, then the weighted ENSC value reduces to the equal division solution; and if $\alpha_i = 1$ for all $i \in N$, the weighted ENSC value coincides with the ENSC value. Moreover, if $\alpha_i = \alpha_j$ for all $i, j \in N$, then the weighted ENSC value degenerates into the α-ENSC value proposed by Sun *et al.* (2017 [15]).

3 Procedural implementation of the weighted ENSC value

Generally, it is assumed that the grand coalition has to form in cooperative situations.[5] The allocation scenario envisaged to compute the weighted ENSC value takes up this idea

[5]This assumption is reasonable in many real-life situations such as, for example, the constitution of an information exchange forum among the firms using the common chemical substance under the new European legislation REACH (Dehez and Tellone, 2013 [4]).

and consists of the following steps.

(i) Choose any TU-game $(N, v) \in G$ and any order $\pi \in \Pi(N)$ to gradually form the grand coalition N.

(ii) Each entering player $i \in N$ such that $\pi^{-1}(i) = 1$ receives his individual worth $v(\{i\})$.

(iii) Each entering player $i \in N$ such that $\pi^{-1}(i) \neq 1$ obtains a part α_i of his contribution b_i^v.

(iv) The residual (positive or negative) $v(S_\pi^i) - v(S_\pi^i \setminus \{i\}) - \alpha_i b_i^v$ brought by player i joining the non-empty coalition $S_\pi^i \setminus \{i\} = \{j \in N | \pi^{-1}(j) < \pi^{-1}(i)\}$ is evenly distributed among the coalition members except player i.

(v) Steps (i)-(iv) determine a payoff vector $(\gamma_i^\pi)_{i \in N} \in R^N$ defined as:

$$\gamma_i^\pi = \begin{cases} v(\{i\}) + \sum_{t=\pi^{-1}(i)+1}^{n} \frac{v(S_\pi^{\pi(t)}) - v(S_\pi^{\pi(t)} \setminus \pi(t)) - \alpha_{\pi(t)} b_{\pi(t)}^v}{t-1}, & \text{if } \pi^{-1}(i) = 1 \\ \alpha_i b_i^v + \sum_{t=\pi^{-1}(i)+1}^{n} \frac{v(S_\pi^{\pi(t)}) - v(S_\pi^{\pi(t)} \setminus \pi(t)) - \alpha_{\pi(t)} b_{\pi(t)}^v}{t-1}, & \text{if } \pi^{-1}(i) \neq 1 \end{cases} \tag{3.1}$$

(vi) The **allocation scenario outcome**, denoted by $\psi(N, v)$, is defined as the average over all the $n!$ orders of all the payoff vectors given by (3.1), i.e.:

$$\psi_i(N, v) = \sum_{\pi \in \Pi(N)} \frac{1}{n!} \gamma_i^\pi, \quad \text{for all } i \in N \tag{3.2}$$

In Step (iii), player i is not the first entrant in the game. His payoff is composed of two parts: one is a fraction α_i of his contribution b_i^v, while the other is the extra residual obtained from the set of his successors $\{\pi(\pi^{-1}(i) + 1), \ldots, \pi(n)\}$. In Step (ii), player i is initially alone in the game so that the first part of his payoff corresponds to his individual worth $v(\{i\})$. Particularly, if $\alpha_i = 0$ for all $i \in N$, then players are purely altruistic and never occupy any gains when joining the game. For the other extreme case where $\alpha_i = 1$ for all $i \in N$, all the players are totally selfish and never share their marginal contributions to the grand coalition with the others.

A three-player example of the allocation scenario

We consider a three-player game $(N, v) \in G$ where $N = \{1, 2, 3\}$ and an order $\pi \in \Pi(N)$ such that $\pi(1) = 3$, $\pi(2) = 1$ and $\pi(3) = 2$. Player 3 first joins the game and obtains his individual worth $v(\{3\})$, with no earnings for the other two players. At this step, players' payoff vector is $(0, 0, v(\{3\}))$. Then, player 1 enters the game. Due to the allocation scenario rules, player 1 receives a part α_1 of his contribution b_1^v, and player 3 obtains the extra residual produced by player 1's entering which is equal to $v(\{1,3\})-v(\{3\})-\alpha_1 b_1^v$. Player 2 still have no earnings and players' payoff vector is given by $(\alpha_1 b_1^v, 0, v(\{1,3\})-v(\{3\})-\alpha_1 b_1^v)$. Finally, player 2 obtains $\alpha_2 b_2^v$ by joining the game and brings the extra residual $v(N)-v(\{1,3\})-\alpha_2 b_2^v$ which is evenly distributed among players 1 and 3. Therefore, players' payoff vector is $(\frac{v(N)-v(\{1,3\})-\alpha_2 b_2^v}{2}, \alpha_2 b_2^v, \frac{v(N)-v(\{1,3\})-\alpha_2 b_2^v}{2})$. Summating the payoffs obtained at the three steps, we achieve $(\gamma_i^\pi)_{i\in N} = (\alpha_1 b_1^v + \frac{1-\alpha_2}{2} b_2^v, \alpha_2 b_2^v, v(\{1,3\}) - \alpha_1 b_1^v + \frac{1-\alpha_2}{2} b_2^v)$. Observe that each player's total payoff not only relies on the order π, but also depends on the selfish level α_i for all $i \in N$.

Our first main result brings to light the relationship between the allocation scenario outcome and the weighted ENSC value.

Theorem 3.1. For every TU-game $(N,v) \in G$, the allocation scenario outcome given by (3.2) coincides with the weighted ENSC value, i.e.:

$$\psi_i(N,v) = ENSC_i^\alpha(N,v), \quad \text{for all } i \in N.$$

Proof. For notational convenience, we denote $\frac{(s-2)!(n-s)!}{n!}$ by t_s, $\frac{(t-1)!(n-t-1)!}{n!}$ by β_t, and $v(S_\pi^{\pi(k)}) - v(S_\pi^{\pi(k)} \backslash \pi(k))$ by $m_{\pi(k)}$. It follows from (3.1) that:

$$\begin{aligned}
\sum_{\pi\in\Pi(N)} \frac{1}{n!}\gamma_i^\pi &= \sum_{\pi:\pi^{-1}(i)=1} \frac{1}{n!}\Big(v(\{i\}) + \sum_{k=\pi^{-1}(i)+1}^{n} \frac{m_{\pi(k)} - \alpha_{\pi(k)} b_{\pi(k)}^v}{k-1}\Big) \\
&\quad + \sum_{\pi:\pi^{-1}(i)\neq 1} \frac{1}{n!}\Big(\alpha_i b_i^v + \sum_{k=\pi^{-1}(i)+1}^{n} \frac{m_{\pi(k)} - \alpha_{\pi(k)} b_{\pi(k)}^v}{k-1}\Big) \\
&= \frac{1}{n} v(\{i\}) + \sum_{\pi:\pi^{-1}(i)=1} \frac{1}{n!} \sum_{k=\pi^{-1}(i)+1}^{n} \frac{m_{\pi(k)} - \alpha_{\pi(k)} b_{\pi(k)}^v}{k-1} \\
&\quad + \frac{n-1}{n}\alpha_i b_i^v + \sum_{\pi:\pi^{-1}(i)\neq 1} \frac{1}{n!} \sum_{k=\pi^{-1}(i)+1}^{n} \frac{m_{\pi(k)} - \alpha_{\pi(k)} b_{\pi(k)}^v}{k-1} \\
&= \frac{1}{n} v(\{i\}) + \frac{n-1}{n}\alpha_i b_i^v + \frac{1}{n!} \sum_{\pi\in\Pi(N)} \sum_{k=\pi^{-1}(i)+1}^{n} \frac{m_{\pi(k)} - \alpha_{\pi(k)} b_{\pi(k)}^v}{k-1} \\
&= \frac{1}{n} v(\{i\}) + \frac{n-1}{n}\alpha_i b_i^v + \sum_{l=\pi(k)\in N\backslash\{i\}} \sum_{S=S_\pi^{\pi(k)}\subseteq N, S\ni i,l} \frac{m_{\pi(k)} - \alpha_{\pi(k)} b_{\pi(k)}^v}{|S|-1} \cdot \frac{(s-1)!(n-s)!}{n!} \\
&= \frac{1}{n} v(\{i\}) + \frac{n-1}{n}\alpha_i b_i^v + \sum_{l\in N\backslash\{i\}} \sum_{S\subseteq N, S\ni i,l} (v(S) - v(S\backslash\{l\}) - \alpha_l b_l^v) \frac{(s-2)!(n-s)!}{n!}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}v(\{i\}) + \frac{n-1}{n}\alpha_i b_i^v + \sum_{l\in N\setminus\{i\}}\sum_{S\subseteq N, S\ni i,l} v(S)t_s - \sum_{l\in N\setminus\{i\}}\sum_{S\subseteq N, S\ni i,l} v(S\setminus\{l\})t_s \\
&\quad - \sum_{l\in N\setminus\{i\}}\sum_{S\subseteq N, S\ni i,l}\alpha_l b_l^v t_s \\
&= \frac{1}{n}v(\{i\}) + \frac{n-1}{n}\alpha_i b_i^v + \sum_{S\subseteq N, S\ni i, |S|\geq 2}\sum_{l\in S\setminus\{i\}} v(S)t_s - \sum_{l\in N\setminus\{i\}}\sum_{T\subseteq N\setminus\{l\}, T\ni i} v(T)\beta_t \\
&\quad - \sum_{l\in N\setminus\{i\}}\sum_{s=2}^{n}\alpha_l b_l^v t_s\binom{n-2}{s-2} \\
&= \frac{1}{n}v(\{i\}) + \frac{n-1}{n}\alpha_i b_i^v + \sum_{S\subseteq N, S\ni i, |S|\geq 2} v(S)(s-1)t_s - \sum_{T\neq N, T\ni i}\sum_{l\notin T} v(T)\beta_t \\
&\quad - \sum_{l\in N\setminus\{i\}}\sum_{s=2}^{n}\alpha_l b_l^v\frac{1}{n(n-1)} \\
&= \frac{1}{n}v(\{i\}) + \frac{n-1}{n}\alpha_i b_i^v + \sum_{S\subseteq N, S\ni i, |S|\geq 2} v(S)\frac{(s-1)!(n-s)!}{n!} \\
&\quad - \sum_{T\neq N, T\ni i} v(T)\frac{(t-1)!(n-t)!}{n!} - \sum_{l\in N\setminus\{i\}}\frac{\alpha_l b_l^v}{n} \\
&= \frac{1}{n}v(\{i\}) + \frac{n-1}{n}\alpha_i b_i^v + \frac{v(N)}{n} - \frac{v(\{i\})}{n} - \sum_{l\in N\setminus\{i\}}\frac{\alpha_l b_l^v}{n} \\
&= \alpha_i b_i^v + \frac{1}{n}\Big(v(N) - \sum_{l\in N}\alpha_l b_l^v\Big),
\end{aligned}
$$

which completes the proof. □

4 Optimization implementation of the weighted ENSC value

The optimization implementation of solution concepts has received considerable attention over the past few years. By way of examples, we can cite Schmeidler's nucleolus (Schmeidler, 1969 [12]) and Ruiz's least square pre-nucleolus (Ruiz *et al.*, 1996 [11]). Following in the footsteps of these works, we introduce an alternative measure of players' dissatisfaction and prove that the weighted ENSC value lexicographically minimizes the associated complaint vector.

For notational convenience, we will write $\sum_{i\in S} x_i$ as $x(S)$. Given a TU-game $(N, v) \in G$, a payoff vector $x \in R^N$, and players' selfishness levels $\alpha_i \in [0, 1]$, $i \in N$, **the selfish complaint** of coalition S at x, denoted by $\hat{e}(S, x)$, is defined as:

$$\hat{e}(S,x) = \sum_{i\in S} \alpha_i b_i^v - x(S), \text{ for all } S \subseteq N \tag{4.1}$$

Thereby, the coalition members hope to obtain at least a part of their marginal contributions to the grand coalition depending on their selfisness levels and express a positive amount of complaint if their payoffs are below these expectations. For every payoff vector $x \in R^N$, let $\theta(x)$ be the 2^n-tuple whose components are the selfish complaints of coalitions S at x arranged in non-increasing order, i.e.:

$$\theta_i(x) \geq \theta_j(x), \quad \text{if} \quad 1 \leq i \leq j \leq 2^n.$$

Given $x, y \in R^{2^n}$, we say that x is lexicographically smaller than or equal to y, denoted by $x \leq_L y$, if $x = y$ or if there exists $t \in \{1, 2, \ldots, 2^n\}$ such that $x_k = y_k$ for all $k \in \{1, 2, \ldots, t-1\}$ and $x_t < y_t$. The **optimal compromise value** corresponds to the unique pre-imputation $x^{OC} \in I^*(N,v)$ and whose complaint vector $\theta(x^{OC})$ given by (4.1) possesses the smallest lexicographic order $\leq_L$, i.e.:

$$\theta(x^{OC}) \leq_L \theta(x), \quad \text{for all } x \in I^*(N,v).$$

The following result explores the relationship between the optimal compromise value and players' selfishness.

Theorem 4.1. For every TU-game $(N,v) \in G$, it holds that:

(i) If $\sum_{i\in N} \alpha_i b_i^v \geq v(N)$, then the optimal compromise value x^{OC} is bounded above by $(\alpha_i b_i^v)_{i\in N}$, i.e., $x_i^{OC} \leq \alpha_i b_i^v$ for all $i \in N$;

(ii) If $\sum_{i\in N} \alpha_i b_i^v < v(N)$, then the optimal compromise value x^{OC} is bounded below by $(\alpha_i b_i^v)_{i\in N}$, i.e., $x_i^{OC} \geq \alpha_i b_i^v$ for all $i \in N$.

Proof. (i) For the sake of contradiction, suppose that $x_i^{OC} \leq \alpha_i b_i^v$ does not hold for all $i \in N$.

Case 1: Assume that $x_i^{OC} \geq \alpha_i b_i^v$ holds for all $i \in N$ and there exists at least one player $j \in N$ such that $x_j^{OC} > \alpha_j b_j^v$. Then, it holds that:

$$\sum_{i\in N} x_i^{OC} > \sum_{i\in N} \alpha_i b_i^v \geq v(N),$$

which contradicts $x^{OC} \in I^*(N,v)$.

Case 2: Assume that there exist at least two players $l, m \in N$ such that $x_l^{OC} < \alpha_l b_l^v$ and $x_m^{OC} > \alpha_m b_m^v$. Denote $\varepsilon = \min\{\alpha_l b_l^v - x_l^{OC}, x_m^{OC} - \alpha_m b_m^v\} > 0$ and define the new payoff vector $x^* \in I^*(N,v)$ as follows:

$$x_i^* = \begin{cases} x_i^{OC} & \text{for all } i \notin \{l,m\}; \\ x_l^{OC} + \varepsilon & \text{if } i = l; \\ x_m^{OC} - \varepsilon & \text{if } i = m. \end{cases}$$

Then, it holds that:

$$\theta_1(x^{OC}) = (\alpha_l b_l^v - x_l^{OC}) + \sum_{i \in N \setminus \{l\}: \alpha_i b_i^v > x_i^{OC}} (\alpha_i b_i^v - x_i^{OC}),$$

and

$$\theta_1(x^*) = (\alpha_l b_l^v - (x_l^{OC} + \varepsilon)) + \sum_{i \in N \setminus \{l\}: \alpha_i b_i^v > x_i^{OC}} (\alpha_i b_i^v - x_i^{OC}),$$

where the second equality holds because $\alpha_l b_l^v - (x_l^{OC} + \varepsilon) \geq 0$. Thus, we have $\theta(x^{OC}) >_L \theta(x^*)$, a contradiction.
Therefore, we conclude that $x_i^{OC} \leq \alpha_i b_i^v$ for all $i \in N$.
(ii) The proof is similar to that of (i). $\square$

The next lemma brings to light the changes of coalitions' selfish complaints at different payoff vectors.

Lemma 4.2. Given a TU-game $(N, v) \in G$, a payoff vector $x \in I^*(N, v)$, two players $l, m \in N$ such that $\alpha_l b_l^v - x_l > \alpha_m b_m^v - x_m \geq 0$, we denote $\varepsilon = \frac{\alpha_l b_l^v - x_l - (\alpha_m b_m^v - x_m)}{2}$. Then, for the new payoff vector $x^* \in I^*(N, v)$ defined as:

$$x_i^* = \begin{cases} x_i & \text{for all } i \notin \{l, m\}; \\ x_l + \varepsilon & \text{if } i = l; \\ x_m - \varepsilon & \text{if } i = m. \end{cases}$$

the following five statements hold:

(i) For any $S \subseteq N$ such that $S \not\ni l, m$, we have $\hat{e}(S, x^*) = \hat{e}(S, x)$;

(ii) For any $S \subseteq N$ such that $S \ni l, m$, we have $\hat{e}(S, x^*) = \hat{e}(S, x)$;

(iii) For any $S \subseteq N$ such that $S \ni l$ and $S \not\ni m$, we have $\hat{e}(S, x^*) < \hat{e}(S, x)$;

(iv) For any $S \subseteq N$ such that $S \not\ni l$ and $S \ni m$, it is redundant to consider $\hat{e}(S, x)$ and $\hat{e}(S, x^*)$ when comparing $\theta(x)$ and $\theta(x^*)$;

(v) It holds that $\theta(x) >_L \theta(x^*)$.

Proof. (i) It is trivial that for any $S \subseteq N$ such that $S \not\ni l, m$, $\hat{e}(S, x) = \hat{e}(S, x^*)$ holds.
(ii) For any $S \subseteq N$ such that $S \ni l, m$, it holds that:

$$
\begin{aligned}
\hat{e}(S,x^*) &= \sum_{i\in S}\alpha_i b_i^v - x^*(S)\\
&= (\alpha_l b_l^v - x_l^*) + (\alpha_m b_m^v - x_m^*) + \sum_{i\in S\backslash\{l,m\}}\alpha_i b_i^v - x^*(S\backslash\{l,m\})\\
&= (\alpha_l b_l^v - (x_l+\varepsilon)) + (\alpha_m b_m^v - (x_m-\varepsilon)) + \sum_{i\in S\backslash\{l,m\}}\alpha_i b_i^v - x(S\backslash\{l,m\})\\
&= \sum_{i\in S}\alpha_i b_i^v - x(S)\\
&= \hat{e}(S,x).
\end{aligned}
$$

(iii) For any $S \subseteq N$ such that $S \ni l$ and $S \not\ni m$, it holds that:

$$
\begin{aligned}
\hat{e}(S,x^*) &= \sum_{i\in S}\alpha_i b_i^v - x^*(S)\\
&= (\alpha_l b_l^v - x_l^*) + \sum_{i\in S\backslash\{l\}}\alpha_i b_i^v - x^*(S\backslash\{l\})\\
&= (\alpha_l b_l^v - (x_l+\varepsilon)) + \sum_{i\in S\backslash\{l\}}\alpha_i b_i^v - x(S\backslash\{l\})\\
&= \sum_{i\in S}\alpha_i b_i^v - x(S) - \varepsilon\\
&= \hat{e}(S,x) - \varepsilon\\
&< \hat{e}(S,x),
\end{aligned}
$$

where the last inequality holds because $\varepsilon > 0$.
(iv) For any $S \subseteq N$ such that $S \not\ni l$ and $S \ni m$, it holds that:

$$
\begin{aligned}
\hat{e}(S,x) &= \sum_{i\in S}\alpha_i b_i^v - x(S)\\
&= (\alpha_m b_m^v - x_m) + \sum_{i\in S\backslash\{m\}}\alpha_i b_i^v - x(S\backslash\{m\})\\
&< (\alpha_l b_l^v - x_l) + \sum_{i\in S\backslash\{m\}}\alpha_i b_i^v - x(S\backslash\{m\})\\
&= \sum_{i\in S\cup\{l\}\backslash\{m\}}\alpha_i b_i^v - x(S\cup\{l\}\backslash\{m\})\\
&= \hat{e}(S\cup\{l\}\backslash\{m\},x).
\end{aligned}
$$

Therefore, for any $S \subseteq N$ such that $S \not\ni l$ and $S \ni m$, there always exists a coalition $S\cup\{l\}\backslash\{m\}$ such that $\hat{e}(S\cup\{l\}\backslash\{m\},x) > \hat{e}(S,x)$. Moreover, observe that $\alpha_l b_l^v - x_l^* = \alpha_m b_m^v - x_m^*$ which implies that:

$$
\begin{aligned}
\hat{e}(S,x^*) &= \sum_{i\in S}\alpha_i b_i^v - x^*(S) \\
&= (\alpha_m b_m^v - x_m^*) + \sum_{i\in S\backslash\{m\}}\alpha_i b_i^v - x^*(S\backslash\{m\}) \\
&= (\alpha_l b_l^v - x_l^*) + \sum_{i\in S\backslash\{m\}}\alpha_i b_i^v - x^*(S\backslash\{m\}) \\
&= \sum_{i\in S\cup\{l\}\backslash\{m\}}\alpha_i b_i^v - x^*(S\cup\{l\}\backslash\{m\}) \\
&= \hat{e}(S\cup\{l\}\backslash\{m\},x^*).
\end{aligned}
$$

It follows from (iii) that $\hat{e}(S\cup\{l\}\backslash\{m\},x^*) < \hat{e}(S\cup\{l\}\backslash\{m\},x)$. Thus, it is redundant to consider $\hat{e}(S,x)$ and $\hat{e}(S,x^*)$ when comparing $\theta(x)$ and $\theta(x^*)$.
(v) The proof of this statement directly follows from (i),(ii),(iii) and (iv). $\square$

Our second main result provides an optimization characterization of the weighted ENSC value.

Theorem 4.3. For every TU-game $(N,v)\in G$ such that $\sum_{i\in N}\alpha_i b_i^v \geq v(N)$, it holds that:

(i) $\alpha_i b_i^v - x_i^{OC} = \alpha_j b_j^v - x_j^{OC}$ for all $i,j\in N$;

(ii) $x_i^{OC} = ENSC_i^{\alpha}(N,v)$ for all $i\in N$.

Proof. (i) For the sake of contradiction, assume that there exist $i,j\in N$ such that $\alpha_i b_i^v - x_i^{OC} \neq \alpha_j b_j^v - x_j^{OC}$. Without loss of generality, let $\alpha_i b_i^v - x_i^{OC} > \alpha_j b_j^v - x_j^{OC}$. It follows from (i) of Theorem 4.1 and (v) of Lemma 4.2 that there exists $x^*\in I^*(N,v)$ such that $\theta(x^{OC}) >_L \theta(x^*)$, a contradiction.
(ii) The proof of this statement directly follows from (i) and $x^{OC}\in I^*(N,v)$. $\square$

5 Conclusion

In this article, we have implemented the weighted ENSC value throughout two approaches: the first one describes a procedural implementation based on a new allocation scenario and the second one provides an optimization implementation by means of players' selfish complaints. The study of other interesting values using similar techniques is left for future research.

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