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Documents de travail GREDEG
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GREDEG WP No. 2018-15
https://ideas.repec.org/s/gre/wpaper.html

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Axiomatic Foundations of a Unifying Concept of the Core of Games in Effectiveness Form

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June 6, 2018

GREDEG Working Paper No. 2018–15

Abstract

We provide an axiomatic characterization of the core of games in effectiveness form. We point out that the core, whenever it applies to appropriate classes of these games, coincides with a wide variety of prominent stability concepts in social choice and game theory, such as the Condorcet winner, the Nash equilibrium, the pairwise stability, and the stable matchings, among others. Our characterization of the core invokes the axioms of non-emptiness, coalitional unanimity, and Maskin monotonicity together with a principle of independence of irrelevant states, and uses in its proof a holdover property echoing the conventional ancestor property. Taking special cases of this general characterization of the core, we derive new characterizations of the previously mentioned stability concepts.

Keywords: Effectiveness function, core, axiomatization, holdover property, consistency principle
JEL Classification: C70, C71

1 Introduction

Most theorists in economics and politics have been occupied in studying a wide variety of stability concepts in social choice and game theory for a century or more. Generally speaking, these stability concepts are mainly founded on the idea that given some prevailing state, individuals possess some blocking power to oppose that state when they have an interest to do so. A stable state is then usually understood to mean a state for which no individual or group of individuals has the power to change the status quo by choosing a more desirable situation. This arises, for example, in a general equilibrium of markets where economic agents on both the demand and supply sides do not have any incentive to alter their consumption or production decisions at the

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given market price. In the same vein, elections in political systems rely on voting rules (quorum, majority, etc.) that allow some coalitions of voters to impose their chosen candidate on the entire society. In like manner, equilibrium concepts for non-cooperative games (Nash equilibrium, subgame perfect equilibrium, etc.) recommend a state robust to deviations in strategy in both static and dynamic settings. Likewise, many solution concepts for coalitional games (core, stable set, etc.) stress cooperative agreements on utility allocation that no coalition would contest.

In this article, we consider the general framework of games in effectiveness form (henceforth e-form games), first introduced by Rosenthal (1972), which encompasses a vast range of contexts, including voting problems, normal form games, network problems, and matching models, among others. The canonical e-form game has the following features. A set N of players is endowed with preferences over a set A of states. Players are mutually aware of each other's preferences, can form coalitions, and sign binding agreements to oppose a given state. In addition, the blocking power distribution among coalitions is described by an effectiveness function that, given a prevailing state a in A, specifies for every coalition S of players and subset $B \subseteq A$ of states whether or not S is effective for B, i.e. coalition S is effective if it can force all players to move from state a to some state in B. Without going into the details, this way of defining the effectivity of coalitions is similar to the inducement correspondence introduced by Greenberg (1990) and is more general than the notions of the effectivity function and effectiveness relation respectively proposed by Moulin and Peleg (1982) and Chew (1994). The effectiveness function also corresponds to a special case of the local effectivity function suggested by Abdou and Keiding (2003).

Since players can behave cooperatively to oppose a given state, the solution concept we consider is a version of the core of e-form games (see Rosenthal, 1972). A state a is core-stable if there is no coalition S of players and subset B of states for which S is effective for B at a and in which every player in S strictly prefers every state in B to a. The most remarkable feature of the core lies in the fact that a wide variety of prominent stability concepts in social choice and game theory, such as the Condorcet winner, the Nash equilibrium, the pairwise stability, and the stable matchings, among others, coincide with the core applied to some classes of e-form games by means of an appropriate effectiveness function (Propositions 3.1, 3.2, 3.3, and 6.5). More precisely, by fixing what constitutes the blocking power of coalitions, we can express these stability concepts in terms of the core for a suitable class of e-form games.

Despite the diversity of existing stability concepts such as those just mentioned, very little is known about the properties that unify them. To address this issue, we propose to axiomatically characterize the core on a vast range of classes of e-form games. Formally, the core is a correspondence that associates each e-form game with a (possibly empty) subset of core-stable states. Perhaps unexpectedly, the core is characterized on a wide range of classes of solvable e-form games, i.e. those with a non-empty core, by a set of four axioms, which are reasonably weak and intuitive: non-emptiness, coalitional unanimity, Maskin monotonicity, and independence of irrelevant states (Theorem 4.4). The non-emptiness axiom obviously requires a solution to contain at least one state. Coalitional unanimity establishes that if a state a is selected for an e-form game, then

a must belong to any unanimously best set of states B for players in some coalition S effective for B at a. If a state a is selected for an e-form game, then Maskin monotonicity asserts that it is also selected in an e-form game where a has (weakly) risen in the preference rankings of all players. Independence of irrelevant states specifies that if a state is selected for an e-form game, then it remains selected even when non-selected states are withdrawn from the game. This latter axiom is in line with other principles of independence widely used in characterizations of game-theoretic solutions (see Nash, 1950a; Arrow, 1950; Chernoff, 1954; Sen, 1969, 1993). For some classes of e-form games such as, for example, those derived from network problems, states cannot be withdrawn without removing players associated with them. Our principle of independence permits removing such players when necessary. We first show that if a solution is coalitionally unanimous and Maskin monotonic, then it is a subsolution of the core (Proposition 4.1). Then, we prove that if a subsolution of the core is non-empty and complies with the independence of irrelevant states, then it is equal to the core (Lemma 4.3), provided the class of e-form games satisfies a new property, called the holdover property, which plays a key role in the proof of this statement. This property echoes the conventional ancestor property and specifies that, given a state a in the core of an e-form game, it is always possible to introduce additional states (and their associated players when necessary) in such a way that the core of the new augmented e-form game only contains state a. This methodology constitutes an alternative to the use of the so-called bracing lemma, which is a typical consistency result for many game-theoretic models (see Thomson, 2011). The complementarity of these two approaches is highlighted in Subsection 6.2. Using the building blocks leading up to our axiomatic characterization of the core (Theorem 4.4), we provide new axiomatic characterizations of the Condorcet winner, the Nash equilibrium, and the pairwise stability. This mainly consists in reformulating our general axioms for specific classes of e-form games underlying these stability concepts and showing that these classes satisfy the holdover property. Proposition 5.2 characterizes the Condorcet winner by the axioms of non-emptiness, majority criterion, and Maskin monotonicity. In Proposition 5.4, we invoke the axioms of non-emptiness, weak individual rationality, Maskin monotonicity, and independence of irrelevant strategies to characterize the Nash equilibrium. Proposition 5.6 shows that the pairwise stability is characterized by non-emptiness, one-sided undesirable link, two-sided desirable link, Maskin monotonicity, and independence of irrelevant states. For each of these classes of e-form games, the axioms of majority criterion, weak individual rationality, and one-sided undesirable link, together with two-sided desirable link are derived from coalitional unanimity. Furthermore, the axiom of independence of irrelevant strategies follows from the axiom of independence of irrelevant states. As far as we know, the Condorcet winner and the pairwise stability have never been characterized axiomatically before. Our characterization of the Nash equilibrium is compared with the existing ones proposed by Peleg and Tijs (1996) and Ray (2000), allowing two other axiomatic characterizations to be established (Proposition 6.2). By invoking the consistency principle and applying the bracing lemma in the framework of e-form games, Proposition 6.7 provides a new characterization of the stable matchings by the axioms of non-emptiness, soulmates (derived from coalitional unanimity),

Maskin monotonicity, and consistency.
The rest of the article is organized as follows. Section 2 introduces the framework of e-form games and the related concept of the core. Specific classes of e-form games for which the core coincides with existing stability concepts are constructed in Section 3. Section 4 presents the main axiomatic characterization of the core. Section 5 contains the specific characterizations of the Condorcet winner, the Nash equilibrium and the pairwise stability. The consistency principle for e-form games is discussed in Section 6 and used to characterize the Nash equilibrium and the stable matchings. All proofs appear in the Appendix.

2 Preliminaries

2.1 Games in effectiveness form

Let $\mathcal{N}$ and $\mathcal{A}$ respectively be fixed and infinite **sets of players and states**. Given a finite subset of players $N \in \mathcal{P}_0(\mathcal{N})$, called a **coalition**, and a subset of states $A \in \mathcal{P}_0(\mathcal{A})$,[1] an **effectiveness function** on (N, A) is a family $E := (E_a, a \in A)$, where for all $a \in A$, $E_a : \mathcal{P}(N) \longrightarrow \mathcal{P}(\mathcal{P}_0(A))$ is such that $E_a(\emptyset) = \emptyset$.[2] As a general interpretation, $B \in E_a(S)$ means that when the current state is $a \in A$, coalition S can force the outcome of the e-form game to be in B. Each player i has a complete and transitive **preference relation** $\succeq_i$ over the set A of states. We will sometimes assume that the preference relation is also antisymmetric, depending on the context in which we will work in, such as voting problems. We respectively denote $\succ_i$ and $\sim_i$ as the asymmetric and symmetric parts of $\succeq_i$. For $a, b \in A$, $a \succ_i b$ means that player i strictly prefers a to b, and $a \sim_i b$ means that player i is indifferent between a and b. Let $\succeq = (\succeq_i)_{i \in N}$ denote the **preference profile** of all players over A.
A **game in effectiveness form (e-form game)** is a tuple $\Gamma = (N, A, E, \succeq)$ consisting of a coalition $N \in \mathcal{P}_0(\mathcal{N})$ of players, a set $A \in \mathcal{P}_0(\mathcal{A})$ of states, an effectiveness function $E := (E_a, a \in A)$, and a preference profile $\succeq$ over A. We denote any **generic class of e-form games** by $\mathcal{K}$. Throughout this article, we assume that if $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ for some preference profile $\succeq$ over A then $\Gamma' = (N, A, E, \succeq') \in \mathcal{K}$ for all preference profiles $\succeq'$ over A.

2.2 The solution concept of the core

A **solution** on $\mathcal{K}$ is a correspondence φ that associates each e-form game $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ with a (possibly empty) subset $\varphi(\Gamma) \in \mathcal{P}_0(A)$ of states. A **subsolution** of φ on $\mathcal{K}$ is a correspondence ψ associating each e-form game $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ with a subset $\psi(\Gamma)$ of states in $\varphi(\Gamma)$. A proper subsolution ψ of φ on $\mathcal{K}$ is a subsolution of φ

[1] For a set X, $\mathcal{P}(X)$ denotes the set of all subsets of X, and $\mathcal{P}_0(X)$ denotes the set of all non-empty subsets of X.

[2] Rosenthal (1972) proposed a closed definition of an effectiveness function which, for every coalition $S \in \mathcal{P}(N)$, maps, independent of N, every state $a \in A$ into a collection of subsets of A. Our definition is inspired by that of the effectivity function proposed by Moulin and Peleg (1982), which assigns, independent of the current state, a collection of subsets of A to every coalition $S \in \mathcal{P}(N)$.

on $\mathcal{K}$ such that $\psi \neq \varphi$.
We now introduce the appropriate notion of objection which will be useful for the definition of the core. For $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$, an **objection** against a state $a \in A$ is a pair $(S, B) \in \mathcal{P}_0(N) \times \mathcal{P}_0(A)$ such that:

(i) $B \in E_a(S)$;

(ii) for all $b \in B$ and all $i \in S$, $b \succ_i a$ holds.

Formally, the **core** of $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$, denoted by $C(\Gamma)$, is defined to be the set of states against which there exists no objection:

$$C(\Gamma) = \{a \in A : \text{there is no objection } (S, B) \text{ against } a\}.$$

An e-form game $\Gamma \in \mathcal{K}$ is **solvable** if it admits a non-empty core. The **set of solvable e-form games** over the class $\mathcal{K}$ is denoted by $\mathcal{K}_C$.

3 Core representations of typical stability concepts

The interpretation of the core may differ from one class of e-form games to another. In this section, we express some prominent stability concepts in terms of the core in a suitable class of e-form games.

3.1 Voting theory, effectivity functions, and the Condorcet winner

The first class of e-form games concerns the selection of a committee in the theory of voting. A **voting problem** is a tuple $(N, A, \succeq)$, where N is a **set of voters**, A is a **set of candidates**, and $\succeq = (\succeq_i)_{i \in N}$ is a **strict preference profile**, i.e. for each voter i, $\succeq_i$ is a complete, transitive, and antisymmetric preference relation over A. The Condorcet winner is a solution concept that is frequently viewed as a natural desideratum for voting problems. For two different candidates a and b, let $P(a, b)$ denote the number of voters who strictly prefer a to b, so that $P(a, b) + P(b, a) = |N|$. Candidate a is said to **beat** candidate b if $P(a, b) > P(b, a)$, or equivalently $P(a, b) > |N|/2$. For a voting problem $(N, A, \succeq)$, the **Condorcet winner** is the unique candidate who beats any other candidate in a head-to-head competition whenever it exists.
In the theory of voting, effectivity functions (Moulin and Peleg, 1982) smoothly describe the allocation of decision power among various coalitions of voters. In this framework, such coalitional power is independent of the chosen candidate, i.e. for all $a, b \in A$ and all $S \in \mathcal{P}(N)$, $E_a(S) = E_b(S)$.[3]
More specifically, the Condorcet winner allows any coalition with a majority to veto any candidate whenever its members agree to do so. As a consequence, independently of the chosen candidate, the associated effectiveness function $E^{CW} := (E_a^{CW}, a \in A)$ must give veto power to a coaliton of size at least $|N|/2$:

[3]Moulin and Peleg (1982) identified a class of additive effectivity functions yielding a non-empty core for all preference profiles.

$$\forall a \in A,\ \forall S \in \mathcal{P}_0(N),\ E_a^{CW}(S) = \begin{cases} \{\{b\}, b \in A\} & \text{if } |S| \geq |N|/2; \\ \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathcal{K}^{CW}$ be the class of e-form games $\Gamma = (N, A, E^{CW}, \succeq)$.

Proposition 3.1. A candidate a is the Condorcet winner for $(N, A, \succeq)$ if and only if $C(\Gamma) = \{a\}$, where $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}^{CW}$.

Since the proof of this result is fairly straightforward, it is omitted. It is worth noting that any effectiveness function gives rise to an equivalent core representation of the Condorcet winner if $|S| \geq |N|/2$, $\{b\} \in E_a(S)$, for all $a, b \in A$, and $E_a(S) = \emptyset$ otherwise.

3.2 Normal form games and Nash equilibrium

As a second class of e-form games, we are concerned with normal form games. A **normal form game** is a tuple $(N, \Sigma_N, \succeq)$ where N is a finite **set of players**, $(\Sigma_i)_{i \in N}$ are the **sets of strategies**, and $\succeq = (\succeq_i)_{i \in N}$ are complete and transitive **preference relations** over the **set of strategy profiles** $\Sigma_N = \prod_{i \in N} \Sigma_i$. For $S \in \mathcal{P}_0(N)$, we define the cartesian product $\Sigma_S = \prod_{i \in S} \Sigma_i$ and for $\sigma \in \Sigma_N$, we denote the projection of σ on Σ_S by σ_S. We say that $\sigma^* \in \Sigma_N$ is a **Nash equilibrium** (Nash, 1950b) of $(N, \Sigma, \succeq)$ if:

$$\forall i \in N,\ \forall \sigma_i \in \Sigma_i,\ \sigma^* \succeq_i (\sigma_i, \sigma^*_{N \setminus \{i\}}).$$

In the framework of e-form games, the set Σ_N of strategy profiles can be identified with the set A of states. To define the associated effectiveness function $E^{NE} := (E_\sigma^{NE}, \sigma \in \Sigma_N)$, only individual players are likely to oppose a state by a deviation in strategy:

$$\forall \sigma \in \Sigma_N,\ \forall S \in \mathcal{P}_0(N),\ E_\sigma^{NE}(S) = \begin{cases} \{\{(\sigma_i', \sigma_{N \setminus \{i\}})\} : \sigma_i' \in \Sigma_i\} & \text{if } S = \{i\}; \\ \\ \emptyset & \text{otherwise.} \end{cases}$$

We denote $\mathcal{K}^{NE}$ as the class of e-form games $\Gamma = (N, \Sigma_N, E^{NE}, \succeq)$.

Proposition 3.2. The set of Nash equilibria of $(N, \Sigma_N, \succeq)$ coincides with $C(\Gamma)$, where $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}$.

The proof of this result is straightforward and thus omitted.

3.3 Network formation and pairwise stability

The third class of e-form games concerns models of network formation (Jackson and Wolinsky, 1996). Given a finite set N of players, a **network** on N is defined as a **set of edges** g that describes pairwise relations among these players. Two players i and

j are **directly connected** in g if and only if $\{i,j\} \in g$. For simplicity, we will write $ij \in g$ if nodes i and j are directly connected, and $ij \notin g$ otherwise. The **set of all networks** on N is then $\{g : g \subseteq g^N\}$, where g^N is the complete network. A preference profile is given by $\succeq = (\succeq_i)_{i\in N}$, where for each player $i \in N$, $\succeq_i$ is a complete and transitive preference relation over the set $\{g : g \subseteq g^N\}$. A **network problem** is a tuple $(N, \{g : g \subseteq g^N\}, \succeq)$. We denote $g + ij$ the network obtained from network g by adding link ij and denote $g - ij$ as the network obtained from g by deleting link ij. A network g is said to be **pairwise stable** (Jackson and Wolinsky, 1996) if the two following conditions hold:

(i) for all $ij \in g$, $g \succeq_i g - ij$ and $g \succeq_j g - ij$;

(ii) for all $ij \notin g$, $g + ij \succ_i g$ implies $g \succeq_j g + ij$.[4]

When considering the framework of e-form games, we identify A with the set $\{g : g \subseteq g^N\}$. The notion of pairwise stability requires that link deletion be one-sided and linked addition be two-sided. As a consequence, the associated effectiveness function $E^{PS} := (E^{PS}_g, g \subseteq g^N)$ only gives blocking power to coalitions of size one or two. First, one-sided link deletion allows any player to delete any of its links:

$$\forall g \subseteq g^N, \forall i \in N, E^{PS}_g(\{i\}) = \{\{g - ij\} : ij \in g\}.$$

Second, two-sided link addition allows any pair of players that are currently not-linked to form a link between themselves:

$$\forall g \subseteq g^N, \forall i,j \in N : i \neq j, E^{PS}_g(\{i,j\}) = \begin{cases} \{\{g + ij\}\} & \text{if } ij \notin g; \\ \\ \emptyset & \text{otherwise.} \end{cases}$$

Third, for $S \in \mathcal{P}_0(N)$ such that $|S| \geq 3$, and for all $g \subseteq g^N$, we assume that $E^{PS}_g(S) = \emptyset$.[5] We denote $\mathcal{K}^{PS}$ as the class of e-form games $\Gamma = (N, A, E^{PS}, \succeq)$.

Proposition 3.3. The set of pairwise stable networks of $(N, \{g : g \subseteq g^N\}, \succeq)$ coincides with $C(\Gamma)$, where $\Gamma = (N, A, E^{PS}, \succeq) \in \mathcal{K}^{PS}$.

Since the proof of this result is fairly straightforward, we omit it.

4 Axiomatic characterizations of the core

Let φ be a solution on $\mathcal{K}$. Maskin monotonicity (Maskin, 1999) is the first axiom of a solution we introduce. It is one of the key concepts in implementation theory but is

[4] Jackson and Wolinsky (1996) proposed a stronger version of pairwise stability requiring that adding a link makes one deviating player strictly better off and the other one weakly better off. The notion introduced here was also discussed by Jackson and Wolinsky (1996) and is largely used in the literature.

[5] Other models of network formation allowing the creation of more than one link at a time by coalitions of arbitrary size (see Dutta and Mutuswami, 1997; Jackson and van den Nouweland, 2005) can be represented by means of appropriate effectiveness functions.

also desirable in and of itself. Roughly speaking, this axiom requires that if a state is chosen in an e-form game, then it is also chosen in an e-form game resulting from a Maskin monotonic transformation. This essentially means that the state has (weakly) improved in the preference rankings of all players. We say that a preference profile $\succeq'$ is a **Maskin monotonic transformation** of a preference profile $\succeq$ at $a \in A$ if any state that is ranked below a under $\succeq_i$ is also ranked below a under $\succeq_i'$, i.e. for all $i \in N$, $L_i(\succeq_i, a) \subseteq L_i(\succeq_i', a)$, where $L_i(\succeq_i, a) = \{b \in A : a \succeq_i b\}$.

Maskin monotonicity (MM). A solution φ is **Maskin monotonic** on $\mathcal{K}_0 \subseteq \mathcal{K}$ if for all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}_0$, all $\Gamma' = (N, A, E, \succeq') \in \mathcal{K}$, and all $a \in \varphi(\Gamma)$, with $\succeq'$ a Maskin monotonic transformation of $\succeq$ at a, then $a \in \varphi(\Gamma')$.

When $\varphi = C$ and $\mathcal{K}_0 = \mathcal{K}_C$, note that both e-form games Γ and Γ' belong to $\mathcal{K}_C$ in **MM**.

The next axiom generalizes the well-known unanimity property. We say that $B \in \mathcal{P}_0(A)$ is **unanimously prefered** by $S \in \mathcal{P}_0(N)$ with respect to $\succeq$ if for all $b \in B$, all $c \in A \backslash B$, and all $i \in S$, it then holds that $b \succ_i c$.

Coalitional unanimity (CU). A solution φ is **coalitionally unanimous** on $\mathcal{K}_0 \subseteq \mathcal{K}$ if the following property holds: for all $\Gamma \in \mathcal{K}_0$ and all $a \in \varphi(\Gamma)$, if there exists (S, B) such that $B \in E_a(S)$ and B is unanimously prefered by S, then $a \in \varphi(\Gamma) \cap B$.

CU ensures that a state a in the solution cannot be outside a set B unanimously prefered by a coalition S that is effective for B at a.

Proposition 4.1. Let $\mathcal{K}$ be a class of e-form games. If φ satisfies **CU** and **MM** on $\mathcal{K}$, then φ is a subsolution of the core.

The statement of Proposition 4.1 implies that if an e-form game Γ has an empty core, then it is not possible to find another non-empty solution on a class containing Γ which satisfies both **CU** and **MM**.

The third axiom of non-emptiness is standard and requires that a solution contains at least one state.

Non-emptiness (NE). A solution φ is **non-empty valued** on $\mathcal{K}_0 \subseteq \mathcal{K}$ if for all $\Gamma \in \mathcal{K}_0$, $\varphi(\Gamma) \neq \emptyset$.

The following result can be immediately deduced from Proposition 4.1.

Proposition 4.2. Let $\mathcal{K}$ be a class of e-form games. If $\mathcal{K}$ is such that for all $\Gamma \in \mathcal{K}_C$, $|C(\Gamma)| = 1$, then the core is the unique solution that satisfies **NE**, **CU**, and **MM** on $\mathcal{K}_C$.

This result is relevant to characterize the core in some specific environments such as normal form games that admit a unique Nash equilibrium (e.g. normal form linear oligopoly games) or the Condorcet winner in voting problems (see Section 5.1).

The fourth axiom we consider is a principle of independence of irrelevant states, which concerns population and state withdrawal in an e-form game $\Gamma = (N, A, E, \succeq)$. For $N' \in \mathcal{P}_0(N)$ and $A' \in \mathcal{P}_0(A)$, let $E' := \{E'_a,\, a \in A'\}$ be the effectiveness function on (N', A') defined for all $a \in A'$ and all $S \in \mathcal{P}_0(N')$ by $E'_a(S) = \{B \in E_a(S) : B \subseteq A'\}$. Further, $\succeq' = (\succeq'_i)_{i \in N'}$ is the preference profile defined for all $a, b \in A'$ and $i \in N'$ by $a \succeq'_i b$ if and only if $a \succeq_i b$. The **e-form subgame** of Γ on (N', A') is denoted by $\Gamma' = (N', A', E', \succeq')$. Loosely speaking, our axiom of independence of irrelevant states models the following: if a state is selected, then this state remains selected even some when non-selected (irrelevant) states withdraw.

Independence of Irrelevant States (IIS). A solution φ satisfies **independence of irrelevant states** on $\mathcal{K}_0 \subseteq \mathcal{K}$ if the following is true: if $\Gamma = (N, A, E, \succeq) \in \mathcal{K}_0$, $a \in \varphi(\Gamma)$, and $\Gamma' = (N', A', E', \succeq')$ is an e-form subgame of Γ such that $\Gamma' \in \mathcal{K}_0$ and $\varphi(\Gamma) \subseteq A'$, then $a \in \varphi(\Gamma')$.

We point out that **IIS** requires that the e-form subgame Γ' belongs to $\mathcal{K}_0$. On the one hand, if N is fixed for all e-form games in $\mathcal{K}_0$, **IIS** closely resembles the seminal axiom of independence of irrelevant alternatives like Nash (1950a).[6] On the other hand, since it is not always allowed to withdraw some states without removing some players for some classes of e-form games,[7] the subsets of states and players in the e-form subgame may be closely connected.

Unlike **CU** and **MM**, we will apply **NE** and **IIS** on classes of e-form games satisfying a new property, called the holdover property, echoing the ancestor property defined in Norde et al. (1996).[8]

Holdover property. A class of e-form games $\mathcal{K}$ satisfies **the holdover property** if for all $\Gamma \in \mathcal{K}$ and all $\overline{a} \in C(\Gamma)$, there exists $\overline{\Gamma} \in \mathcal{K}$ such that:

(i) Γ is an e-form subgame of $\overline{\Gamma}$;

(ii) $C(\overline{\Gamma}) = \{\overline{a}\}$.

When the holdover property holds, it is possible to establish a result that allows **IIS** to be applied in a way similar to the so-called bracing lemma, a typical result based on a principle of consistency to characterize solution concepts in game-theoretic models (see Thomson, 2011 and Section 6.2 for a detailed discussion).

[6] See also, for example, Arrow (1950), Chernoff (1954) and Sen (1969, 1993).

[7] For example, this is the case of the class of e-form games representing network problems.

[8] See also, for example, Peleg et al. (1996) and Peleg and Sudhölter (1997).

Lemma 4.3. Let $\mathcal{K}$ be a class of e-form games satisfying the holdover property. No proper subsolution of the core satisfies **NE** and **IIS** on $\mathcal{K}$.

The following is the main characterization of the core for solvable e-form games.

Theorem 4.4. Let $\mathcal{K}$ be a class of e-form games satisfying the holdover property. The core is the unique solution satisfying **NE**, **CU**, **MM**, and **IIS** on $\mathcal{K}_C$.

The logical independence of the axioms are discussed in the Appendix.

5 Applications

5.1 Axiomatic characterization of the Condorcet Winner

We propose adapting the axioms of Section 4 to the class of e-form games $\mathcal{K}^{CW}$ defined in Subsection 3.1 representing voting problems. In this context, **CU** becomes the very standard axiom of the majority criterion stating that if one candidate is preferred by a majority of voters, then that candidate must be the unique chosen candidate.

Majority criterion (MC). A solution φ satisfies the **majority criterion** on $\mathcal{K}_0 \subseteq \mathcal{K}^{CW}$ if the following property holds: for all $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}_0$, if there exists $b \in A$ and $S \in \mathcal{P}_0(N)$ such that $|S| \geq |N|/2$; and further, for all $c \in A \setminus \{b\}$ and all $i \in S$, $b \succ_i c$, then $\varphi(\Gamma) \subseteq \{b\}$.

Proposition 5.1. On any $\mathcal{K}_0 \subseteq \mathcal{K}^{CW}$, **MC** is equivalent to **CU**.

Combining Proposition 4.2 with Proposition 5.1, we obtain the following result.

Proposition 5.2. The Condorcet winner is the unique solution that satisfies **NE**, **MC**, and **MM** on $\mathcal{K}_C^{CW}$.

5.2 Axiomatic characterization of the Nash equilibrium

According to the same line of reasoning, we express the axioms of Section 4 according to the class of e-form games $\mathcal{K}^{NE}$ defined in Subsection 3.2 representing normal form games. As a consequence, **CU** becomes a weak version of the axiom of individual rationality specifying that if the behavior of the players is such that it is possible for one player to reach his favorite strategy profile, then this strategy profile must be selected.

Weak individual rationality (WIR). A solution φ is **weakly individually rational** on $\mathcal{K}_0 \subseteq \mathcal{K}^{NE}$ if the following property holds: for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}_0$ and all $\sigma \in \varphi(\Gamma)$, if there exists $i \in N$ and $\sigma_i' \in \Sigma_i$ such that for all $\sigma'' \in \Sigma_N$ with $\sigma'' \neq (\sigma_i', \sigma_{-i})$, $(\sigma_i', \sigma_{-i}) \succ_i \sigma''$ holds, then $\sigma_i' = \sigma_i$.

In Section 6, we will establish that **WIR** is implied by the two standard axioms of one-person rationality and weak consistency.

Proposition 5.3. On any $\mathcal{K}_0 \subseteq \mathcal{K}^{NE}$, **WIR** is equivalent to **CU**.

The proof of this result is fairly straightforward and thus omitted. The next axiom was first introduced in the context of normal form games by Peleg and Tijs (1996). Let $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}_0 \subseteq \mathcal{K}^{NE}$. For $i \in N$, let $\Sigma'_i \subseteq \Sigma_i$ and $E'^{NE} := \{E'^{NE}_\sigma, \sigma \in \Sigma'_N\}$ be the effectiveness function on (N, Σ'_N) defined for all $\sigma \in \Sigma'_N$ and all $S \in \mathcal{P}_0(N)$ by:

$$E'^{NE}_\sigma(S) = \begin{cases} \{\{(\sigma'_i, \sigma_{N\setminus\{i\}})\} : \sigma'_i \in \Sigma'_i\} & \text{if } S = \{i\}; \\ \\ \emptyset & \text{otherwise.} \end{cases}$$

Independence of irrelevant strategies (IISt). A solution φ satisfies **independence of irrelevant strategies** on $\mathcal{K}_0 \subseteq \mathcal{K}^{NE}$ if the following property holds: if $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}_0$, $\sigma \in \varphi(\Gamma)$, $\sigma_i \in \Sigma'_i \subseteq \Sigma_i$ for all $i \in N$, and $\Gamma' = (N, \Sigma'_N, E'^{NE}, \succeq') \in \mathcal{K}_0$ is an e-form subgame of Γ, then $\sigma \in \varphi(\Gamma')$.

It is immediate that **IIS** implies **IISt** over any subclass of $\mathcal{K}^{NE}$ and that with a fixed set of players N, both axioms are equivalent over any subclass of $\mathcal{K}^{NE}$.

Now for all $i \in N$, the set $\mathcal{S}_i$ is defined as an infinite set of possible strategies associated with player i. For all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}$ and all $i \in N$, we assume that $\Sigma_i \subsetneq \mathcal{S}_i$. We say that a class $\mathcal{K}_0 \subseteq \mathcal{K}^{NE}$ is **extendable** if for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}_0$ and all $i \in N$, there exists $\tilde{\Sigma}_i \subsetneq \mathcal{S}_i$ such that $\Sigma_i \subsetneq \tilde{\Sigma}_i$; and further, if for all preference profiles $\tilde{\succeq}$ over the set $\tilde{\Sigma}_i \times \Sigma_{N\setminus\{i\}}$, $\tilde{\Gamma} = (N, \tilde{\Sigma}_i \times \Sigma_{N\setminus\{i\}}, \tilde{E}^{NE}, \tilde{\succeq}) \in \mathcal{K}_0$, where $\tilde{E}^{NE}$ is the effectiveness function that extends E^{NE} on $\tilde{\Sigma}_i \times \Sigma_{N\setminus\{i\}}$. A wide variety of classes of e-form games in $\mathcal{K}^{NE}$ are extendable such as, for example, those representing normal form games with finite or compact sets of strategies.[9] We now have the material to provide a new axiomatic characterization of the Nash equilibrium using Proposition 4.1 and Lemma 4.3.

Proposition 5.4. Let $\mathcal{K} \subseteq \mathcal{K}^{NE}$ be an extendable class of e-form games such that for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $|N| \geq 2$. The Nash equilibrium is the unique solution that satisfies **NE**, **WIR**, **MM**, and **IISt** on $\mathcal{K}_C$.

5.3 Axiomatic characterization of pairwise stable networks

As in the two previous subsections, we write the axioms of Section 4 according to the class of e-form games $\mathcal{K}^{PS}$ defined in Subsection 3.3 representing network problems. In this setting, **CU** can be decomposed into two new axioms.

One-Sided Undesirable Link (OSUL). A solution φ satisfies **one-sided undesirable link** on $\mathcal{K}_0 \subseteq \mathcal{K}^{PS}$ if the following property holds: for all $\Gamma = (N, A, E^{PS}, \succeq) \in \mathcal{K}_0$

[9] For example, given $y \in]0, +\infty]$, we can observe that the usual class of games where the set of strategies is the interval $[0, x] \subsetneq [0, y[$ is also extendable.

and all $g \in \varphi(\Gamma)$, if there exists $i \in N$ such that $g - ij \succ_i g'$ for some $j \in N \setminus \{i\}$ and all $g' \subseteq g^N$ with $g' \neq g - ij$, then $ij \notin g$.

Two-Sided Desirable Link (TSDL). A solution φ satisfies **two-sided desirable link** on $\mathcal{K}_0 \subseteq \mathcal{K}^{PS}$ if the following property holds: for all $\Gamma = (N, A, E^{PS}, \succeq) \in \mathcal{K}_0$ and all $g \in \varphi(\Gamma)$, if there exists $i, j \in N$ such that $g + ij \succ_k g'$ for all $k \in \{i, j\}$ and all $g' \subseteq g^N$ with $g' \neq g + ij$, then $ij \in g$.

Proposition 5.5. On any $\mathcal{K}_0 \subseteq \mathcal{K}^{PS}$, **OSUL** $\wedge$ **TSDL** is equivalent to **CU**.

The proof of this proposition is omitted since **OSUL** and **TSDL** are directly derived from **CU**. Combining Propositions 4.1 and 5.5 with Lemma 4.3, we obtain the following result.

Proposition 5.6. The pairwise stability is the unique solution that satisfies **NE**, **OSUL**, **TSDL**, **MM** and **IIS** on $\mathcal{K}_C^{PS}$.

To the best of our knowledge, this is the first axiomatic characterization of the set of pairwise stable networks.

6 The consistency principle for e-form games

The axiom of consistency we consider in this section is of fundamental interest in axiomatic theory and has been examined from numerous angles.[10] Throughout this section, we suppose that for all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$, there exists $(A_i)_{i \in N}$ such that $A = \prod_{i \in N} A_i$. For $S \in \mathcal{P}_0(N)$, we define the cartesian product $A_S = \prod_{i \in S} A_i$ and for $B \in \mathcal{P}_0(A)$ (resp. $a \in A$), we denote B_S (resp. a_S) as the projection of B (resp. a) on A_S. Let $\Gamma \in \mathcal{K}$ be an e-form game and $\overline{a} \in A$. The **reduced e-form game of Γ with respect to both $\overline{a}$ and S** is the e-form game $\Gamma^{S,\overline{a}} = (S, A_S, E^{S,\overline{a}}, \succeq') \in \mathcal{K}$, where S is a proper coalition of N, and the effectiveness function $E^{S,\overline{a}}$ is such that for all $a_S \in A_S$ and all $T \in \mathcal{P}_0(S)$,

$$E^{S,\overline{a}}_{a_S}(T) = \{B_S \in \mathcal{P}_0(A_S),\ \exists B \in E_{(a_S, \overline{a}_{N \setminus S})}(T) \text{ such that } B = B_S \times \{\overline{a}_{N \setminus S}\}\};$$

and further, where the preference profile $\succeq' = (\succeq'_i)_{i \in S}$ over the set A_S is defined for all $a_S, a'_S \in A_S$ and $i \in S$ by $a_S \succeq'_i a'_S$ if and only if $(a_S, \overline{a}_{N \setminus S}) \succeq_i (a'_S, \overline{a}_{N \setminus S})$. We say that $\mathcal{K}_0 \subseteq \mathcal{K}$ is a **reduction-closed family** if for all e-form games $\Gamma = (N, A, E, \succeq) \in \mathcal{K}_0$, all strategy profiles $\overline{a} \in A$, and all proper coalitions S of N, we have $\Gamma^{S,\overline{a}} \in \mathcal{K}_0$.

We now have the material to write the consistency axiom in the context of e-form games.

Consistency (CONS). A solution φ satisfies **consistency** on $\mathcal{K}_0 \subseteq \mathcal{K}$ if the following property holds: for all $\Gamma \in \mathcal{K}_0$, all $\overline{a} \in \varphi(\Gamma)$, and all proper coalitions S of N such that $\Gamma^{S,\overline{a}} \in \mathcal{K}_0$, we have $a_S \in \varphi(\Gamma^{S,\overline{a}})$.

[10] For comprehensive surveys on consistency and its applications, the reader is referred to Thomson (1990, 2011).

6.1 Consistency and Nash equilibrium

In the literature, consistency is widely used to characterize the Nash equilibrium and is generally associated with the less normative axiom of converse consitency (see Thomson, 2011, p. 272). Both axioms impose an axiomatic characterization over a class with variable number of players. Proposition 5.4 proposes a new axiomatic characterization which uses neither consistency nor converse consistency and does not impose the variability of the population. In this subsection, we discuss the possible links between Proposition 5.4 and existing axiomatic characterizations of the Nash equilibrium using consistency. In the context of normal form games, consistency states that if $\sigma \in \Sigma_N$ belongs to the solution of a game, then σ_S belongs to the solution of the game restricted to S, obtained by fixing the strategies of players outside S to $\sigma_{N\setminus S}$. Given an e-form game $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}$ and a strategy profile $\overline{\sigma} \in \Sigma_N$, the **reduced e-form game of Γ with respect to both $\overline{\sigma}$ and S** is the e-form game $\Gamma^{S,\overline{\sigma}} = (S, \Sigma_S, E^{NE,S,\bar{\sigma}}, \succeq') \in \mathcal{K}^{NE}$, whereby S is a proper coalition of N, the effectiveness function $E^{NE,S,\bar{\sigma}}$ is such that

$$\forall \sigma_S \in \Sigma_S,\ \forall T \in \mathcal{P}_0(S),\ E^{NE,S,\bar{\sigma}}_{\sigma_S}(T) = \begin{cases} \{\{(\sigma'_i, \sigma_{S\setminus\{i\}})\} : \sigma'_i \in \Sigma_i\} & \text{if } T = \{i\}; \\ \\ \emptyset & \text{otherwise;} \end{cases}$$

and further, where the preference profile $\succeq' = (\succeq'_i)_{i\in S}$ over the set Σ_S is defined for all $\sigma_S, \sigma'_S \in \Sigma_S$ and $i \in S$ by $\sigma_S \succeq'_i \sigma'_S$ if and only if $(\sigma_S, \overline{\sigma}_{N\setminus S}) \succeq_i (\sigma'_S, \overline{\sigma}_{N\setminus S})$. We propose to weaken **CONS** on $\mathcal{K}^{NE}$ as follows.

Weak consistency (WCONS). A solution φ satisfies **weak consistency** on $\mathcal{K}_0 \subseteq \mathcal{K}^{NE}$ if the following property holds: for all $\Gamma \in \mathcal{K}_0$, all $\overline{\sigma} \in \varphi(\Gamma)$, and all $i \in N$ such that $\Gamma^{\{i\},\overline{\sigma}} \in \mathcal{K}_0$, we have $\sigma_i \in \varphi(\Gamma^{\{i\},\overline{\sigma}})$.

Since the pionnering work of Peleg and Tijs (1996), several axiomatic characterizations of the Nash equilibrium or closed concepts in normal form games have been proposed in the literature.[11] Most of them use consistency together with one-person rationality that we adapt for e-form games as follows.

One-person rationality (OPR). A solution φ satisfies **one-person rationality** on $\mathcal{K}_0 \subseteq \mathcal{K}^{NE}$ if the following property holds: if $\Gamma = (\{i\}, \Sigma_i, E^{NE}, \succeq_i) \in \mathcal{K}_0$ is a one-person e-form game, then $\varphi(\Gamma) = \{\sigma_i \in \Sigma_i : \sigma_i \succeq_i \sigma'_i \text{ for all } \sigma'_i \in \Sigma_i\}$.

The following proposition establishes that **OPR** together with **WCONS** imply **WIR**.

Proposition 6.1. Let φ be a solution on $\mathcal{K} \subseteq \mathcal{K}^{NE}$ and $\mathcal{K}_0 \subseteq \mathcal{K}$ be a reduction-closed family. If φ satisfies **OPR** and **WCONS** on $\mathcal{K}_0$, then φ also satisfies **WIR** on $\mathcal{K}_0$.

[11] See, for example, Peleg et al. (1996), Peleg and Sudhölter (1997), Norde et al. (1996) and van Heumen et al. (1996).

To introduce the next axiom defined by Peleg and Tijs (1996), we say that a player $d \in N$ is **dummy** in $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}$ if $|\Sigma_d| = 1$.

Dummy (DUM). A solution φ satisfies the **dummy property** on a reduction-closed family $\mathcal{K}_0 \subseteq \mathcal{K}^{NE}$ if the following property holds: if $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}_0$ and $d \in N$ is dummy, then $\varphi(\Gamma) = \Sigma_d \times \varphi(\Gamma^{N\setminus\{d\},\overline{\sigma}})$, where $\overline{\sigma} \in \Sigma_N$.

Ray (2000) proposed a weaker version of **DUM** that we can adapt too.

Weak dummy (WDUM). A solution φ satisfies the **weak dummy property** on a reduction-closed family $\mathcal{K}_0 \subseteq \mathcal{K}^{NE}$ if the following property holds: if $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}_0$ and $d \in N$ is dummy, then $\varphi(\Gamma) \subseteq \Sigma_d \times \varphi(\Gamma^{N\setminus\{d\},\overline{\sigma}})$, where $\overline{\sigma} \in \Sigma_N$.

We establish two other axiomatic characterizations of the Nash equilibrium which only use standard properties.

Proposition 6.2. Let $\mathcal{K} \subseteq \mathcal{K}^{NE}$ be a class of e-form games, which is both extendable and reduction-closed, and φ be a solution on $\mathcal{K}$. The following three properties are equivalent:

(i) φ is the Nash equilibrium;

(ii) φ satisfies **NE**, **OPR**, **IISt**, and **WCONS** on $\mathcal{K}_C$;

(iii) φ satisfies **NE**, **OPR**, **IISt,** and **WDUM** on $\mathcal{K}_C$.

6.2 Ancestor Property and application to stable matchings

In this subsection, we discuss a standard method to axiomatize game-theoretic solutions with the axiom of consistency. This method was called the "bracing lemma" by Thomson (2011). We argue that this method can be adapted in the framework of e-form games and discuss new axiomatizations on matching theory by using the bracing lemma together with Proposition 4.1.

6.2.1 Ancestor property and bracing lemma

The following property concerns the nature of the classes of e-form games and adapted the ancestor property defined by Norde et al. (1996) and already mentioned in Section 4.

Ancestor property. A class of e-form games $\mathcal{K}$ satisfies **the ancestor property** if for all $\Gamma = (N, A_N, E, \succeq) \in \mathcal{K}$ and all $a^* \in C(\Gamma)$, there exists $\Gamma' = (N', A'_{N'}, E', \succeq') \in \mathcal{K}$ with $N' \supseteq N$ such that:

(i) Γ is the reduced e-form game of Γ' with respect to both a^* and N;

(ii) for all $a' \in C(\Gamma')$, $a'_N = a^*$ holds.

When the ancestor property holds, it is possible to establish a result almost identical to the so-called "bracing lemma" (see Thomson, 2011).

Lemma 6.3. Let $\mathcal{K}$ be a set of e-form games satisfying the ancestor property. No proper subsolution of the core satisfies **NE** and **CONS** on $\mathcal{K}$.

We can deduce the following result from Lemma 6.3.

Theorem 6.4. Let $\mathcal{K}$ be a class of e-form games satisfying the ancestor property. The core is the unique solution satisfying **NE**, **CU**, **MM**, and **CONS** on $\mathcal{K}_C$.

6.2.2 A new characterization of the core of matching models

As an application of Theorem 6.4, we propose a new axiomatization of the core of the two-sided one-to-one matching model introduced by Gale and Shapley (1962). Stable matchings was axiomatized for the first time in the context of one-to-one matching by Sasaki and Toda (1992) and recently generalized by Nizamogullari and Özkal-Sanver (2014) and Klaus (2017). We shall consider a **matching model** consisting of a finite set N of individuals, partitioned in two subgroups with equal numbers of **men** M and **women** W. Each man m (resp. woman w) has a complete, transitive, and antisymmetric strict preference relation P_m (resp. P_w) over the set W (resp. M). A **matching** is a one-to-one mapping $\mu : M \longrightarrow W$. A **matching problem** is given by a tuple $(M, W, (P_m)_{m\in M}, (P_w)_{w\in W})$. A matching μ is said to be **stable** in $(M, W, (P_m)_{m\in M}, (P_w)_{w\in W})$ if there is no pair $(m, w) \in M \times W$ such that $wP_m\mu(m)$ and $mP_w\mu^{-1}(w)$. The deferred acceptance rule of Gale and Shapley (1962) determines a stable matching that Pareto dominates any other stable matching.[12]
In the setting of e-form games, the set A of states consists of all possible matchings μ. The preferences of the players $(\succeq_i)_{i\in M\cup W}$ over the set A are induced by their preferences $((P_m)_{m\in M}, (P_w)_{w\in W})$ over their match partners, i.e. for every $m \in M$, $\mu \succ_m \mu'$ if and only if $\mu(m)P_m\mu'(m)$, and for every $w \in W$, $\mu \succ_w \mu'$ if and only if $\mu^{-1}(w)P_w\mu'^{-1}(w)$. Several effectiveness functions may lead to a core representation of the stable matchings. We propose the following definition which will turns out to be essential for our axiomatic result of the stable matchings. The requirement of matching stability is that any man and woman that are currently not partners can decide to create a link between themselves:

$$\forall \mu \in A,\ \forall (m, w) \in M \times W,\ E^{SM}_{\mu}(\{m, w\}) = \{\{\mu' :\ \mu'(m) = w\}\},$$

and for $S \in \mathcal{P}_0(N)$ such that $|S| \neq 2$, $E^{SM}_{\mu}(S) = \emptyset$ for all $\mu \in A$. We denote $\mathcal{K}^{SM}$ as the class of e-form games $\Gamma = (N, A, E^{SM}, \succeq)$.

Proposition 6.5. The set of stable matchings of $(M, W, (P_m)_{m\in M}, (P_w)_{w\in W})$ coincides with $C(\Gamma)$, where $\Gamma = (N, A, E^{SM}, \succeq) \in \mathcal{K}^{SM}$.

[12] In general assignment problems, the deferred acceptance rule has been characterized by Kojima and Manea (2010).

The proof of this result is fairly straightforward and thus omitted. The main advantage of this core representation is to have both an advantageous adaptation of coalitional unanimity as well as an equivalence between the consistency axiom as defined in Section 6 and the one as usually defined in matching theory (see, for example, Sasaki and Toda, 1992). Indeed, for all $\Gamma \in \mathcal{K}^{SM}$, a reduced e-form game $\Gamma^{S,\bar{\mu}}$ belonging to $\mathcal{K}^{SM}$ has necessarily equal numbers of men and women, which means that there exist $M' \subseteq M$ and $W' \subseteq W$ such that $S = M' \cup W'$ and $|M'| = |W'|$. Following the definition of the reduced e-form game, it is immediate that the associated effectiveness function is such that for all matching $\mu|_{M'} : M' \longrightarrow W'$ and all $T \in \mathcal{P}_0(M' \cup W')$, $E^{SM,S,\bar{\mu}}_{\mu|_{M'}}(T) = \{\{\mu'|_{M'} : \mu'|_{M'}(m') = w'\}\}$ if $T = \{m', w'\}$, $(m', w') \in M' \times W'$, and $E^{SM,S,\bar{\mu}}_{\mu|_{M'}}(T) = \emptyset$ otherwise.
In the context of matchings, **CU** becomes a natural axiom that states that soulmates must be married together, where soulmates are a man and a woman who would prefer to be with each other than any other individual (Leo et al., 2017).

Soulmates (SOUL). A solution φ satisfies the **soulmates property** on $\mathcal{K}_0 \subseteq \mathcal{K}^{SM}$ if the following property holds: for all $\Gamma = (N, A, E^{SM}, \succeq) \in \mathcal{K}_0$ and all $\mu \in \varphi(\Gamma)$, if there exists B such that for all $\mu' \in B$, $\mu'(m) = w$ and B is unanimously prefered by $\{m, w\}$, then $\mu(m) = w$.

Proposition 6.6. On any $\mathcal{K}_0 \subseteq \mathcal{K}^{SM}$, **SOUL** is equivalent to **CU**.

The proof of this result is fairly straightforward and thus omitted. Applying Proposition 4.1 and Lemma 6.3, we obtain the following result.

Proposition 6.7. The set of stable matchings is the unique solution that satisfies **NE**, **SOUL**, **MM** and **CONS** on $\mathcal{K}^{SM}$.

Appendix

PROOF of Proposition 4.1. Consider any solution φ that satisfies **CU** and **MM** on the class $\mathcal{K}$. Assume for the sake of contradiction that $\varphi(\Gamma) \not\subseteq C(\Gamma)$ for some $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$. Hence, there is a state $a \in \varphi(\Gamma)$ such that $a \notin C(\Gamma)$, i.e. there exists an objection (S, B) against a in Γ. We now define the preference profile $\succeq'$ as follows:

(i) for all $i \in S$, we first move, in any order, the set of states B to the top of player i's preferences (if not already the case), i.e. for all $b \in B$ and for all $c \in A\backslash B$, it holds that $b \succ'_i c$. Second, we move state a just below the worst state in B according to player i's preferences, i.e. for all $c \in A\backslash(B \cup \{a\})$, $a \succeq'_i c$. Third, for all $c, d \in A\backslash(B \cup \{a\})$, we assume that $c \succeq'_i d$ if and only if $c \succeq_i d$;

(ii) for all $i \in N\backslash S$, we leave preferences as they are, i.e. for all $i \in N\backslash S$, $\succeq'_i = \succeq_i$.

Observe that for all $i \in N$, $L_i(\succeq_i, a) \subseteq L_i(\succeq'_i, a)$. Hence, $\succeq'$ is a Maskin monotonic transformation of $\succeq$ at a. It follows from **MM** that $a \in \varphi(\Gamma')$ where $\Gamma' = (N, A, E, \succeq')$. Furthermore, by the definition of $\succeq'$, it holds that B is unanimously prefered by S. Thus, it follows from **CU** that $a \in B$, a contradiction. ■

PROOF of Lemma 4.3. Let $\mathcal{K}$ be a class satisfying the holdover property and φ be a proper solution of the core, which is defined on $\mathcal{K}$ and satisfies **NE** and **IIS**. Since φ is a proper solution of the core, there exists $\Gamma \in \mathcal{K}$ and $\overline{a} \in C(\Gamma)$ such that $\overline{a} \notin \varphi(\Gamma)$. By the holdover property, there exists $\overline{\Gamma} \in \mathcal{K}$ such that Γ is an e-form subgame of $\overline{\Gamma}$, where $C(\overline{\Gamma}) = \{\overline{a}\}$. Since φ is a proper subsolution of the core satisfying **NE**, we deduce that $\varphi(\overline{\Gamma}) = \overline{a}$. Since φ satisfies **IIS**, it follows that $\overline{a} \in \varphi(\Gamma)$, a contradiction. ■

PROOF of Theorem 4.4. Let $\mathcal{K}$ be a class of e-form games satisfying the holdover property. From Proposition 4.1 and Lemma 4.3, it suffices to prove that the core defined on $\mathcal{K}$ satisfies **NE**, **CU**, **MM**, and **IIS** on $\mathcal{K}_C$. It is immediate that the core satisfies **NE** and **CU**. Second, we prove that the core satisfies **MM** on $\mathcal{K}_C$. Let $\Gamma = (N, A, E, \succeq) \in \mathcal{K}_C$ and $a \in C(\Gamma)$. Consider any $\Gamma' = (N, A, E, \succeq') \in \mathcal{K}$ such that $\succeq'$ is a Maskin monotonic transformation of $\succeq$ at a. Assume for the sake of contradiction that $a \notin C(\Gamma')$. Then, there exists an objection (S, B) against a in Γ'. Hence, for all $b \in B$ and for all $i \in S$, $b \notin L_i(\succeq'_i, a)$, and so $b \notin L_i(\succeq_i, a)$ since $L_i(\succeq_i, a) \subseteq L_i(\succeq'_i, a)$. This implies that for all $b \in B$ and for all $i \in S$, $b \succ_i a$. Thus, (S, B) is also an objection against a in Γ, contradicting $a \in C(\Gamma)$. Third, we show that the core satisfies **IIS** on $\mathcal{K}_C$. Let $\Gamma = (N, A, E, \succeq) \in \mathcal{K}_C$, $a \in C(\Gamma)$ and $\Gamma' = (N', A', E', \succeq')$ be an e-form subgame of Γ such that $\Gamma' \in \mathcal{K}_C$ and $C(\Gamma) \subseteq A'$. Assume for the sake of contradiction that $a \notin C(\Gamma')$. Then, there exists an objection (S, B) against a in Γ'. Since for every $S \in \mathcal{P}_0(N')$, $E'_a(S) = E_a(S)$, and $\succeq_{N'} = (\succeq_i)_{i \in N'}$, (S, B) is also an objection against a in Γ, there is a contradiction of $a \in C(\Gamma)$. It follows that the core also satisfies **IIS** on $\mathcal{K}_C$. ■

PROOF of Proposition 5.1. Let $\mathcal{K}_0 \subseteq \mathcal{K}^{CW}$ and φ be a solution on $\mathcal{K}^{CW}$.

(i) First, assume that φ satisfies **CU** on $\mathcal{K}_0$. Let $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}_0$ and assume that there exists $b \in A$ and $S \in \mathcal{P}_0(N)$ with $|S| \geq \frac{|N|}{2}$, such that for all $c \in A \setminus \{b\}$ and all $i \in S$, $b \succ_i c$. Then $\{b\}$ is unanimously prefered by S and for all $a \in A$, $\{b\} \in E^{CW}_a(S)$. If $\varphi(\Gamma) = \emptyset$, then there is nothing to prove. If $\varphi(\Gamma) \neq \emptyset$, then there exists $a \in \varphi(\Gamma)$, and by **CU**, $a \in \varphi(\Gamma) \cap \{b\}$. We deduce that $\varphi(\Gamma) = \{b\}$.

(ii) Second, assume that φ satisfies **MC** on $\mathcal{K}_0$. Let $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}_0$ and assume that there exist $a \in \varphi(\Gamma)$ and (S, B) such that $B \in E^{CW}_a(S)$ and B is unanimously prefered by S. By the definition of E^{CW}, we deduce that $|S| \geq \frac{|N|}{2}$ and there exists $b \in A$ such that $B = \{b\}$. As a consequence, for all $i \in S$ and all $c \in A \setminus \{b\}$, we have $b \succ_i c$. By **MC**, $\varphi(\Gamma) \subseteq \{b\}$, and we deduce that $a = b$, so $a \in \varphi(\Gamma) \cap \{b\}$.

This concludes the proof. ■

PROOF of Proposition 5.4. Let $\mathcal{K} \subseteq \mathcal{K}^{NE}$ be an extendable class such that for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $|N| \geq 2$. By Proposition 4.1 and Lemma 4.3, it is sufficient to prove that $\mathcal{K}_C$ restricted to games with at least two players satisfies the holdover property. Let $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}_C$ be an e-form game and i and j denote two distinct players belonging to N. We have seen (see Section 3.2) that the set of Nash equilibria of $(N, \Sigma_N, \succeq)$ is equal to $C(\Gamma)$. Assume that $|C(\Gamma)| \geq 2$ and let σ^* be a member of $C(\Gamma)$. Since $\mathcal{K}$ is extendable, for all $k \in \{i, j\}$, there exist $s_k, t_k \in \mathcal{S}_k \setminus \Sigma_k$ such that for all weak order $\tilde{\succeq}$ on $\tilde{\Sigma}_N$, $\tilde{\Gamma} = (N, \tilde{\Sigma}_N, \tilde{E}^{NE}, \tilde{\succeq}) \in \mathcal{K}$, whereby:

$$\tilde{\Sigma}_N = \tilde{\Sigma}_{\{ij\}} \times \Sigma_{N\setminus\{i,j\}} \text{ and } \tilde{\Sigma}_k \supseteq \Sigma_k \cup \{s_k, t_k\} \text{ for all } k \in \{i, j\}.$$

Let $(\tilde{\succeq}_\ell)_{\ell \in N}$ be the preference profile on $\tilde{\Sigma}_N$ defined as follows:

Rule 1. $\forall \ell \in N$, $\forall k, k' \in \{i, j\}$, $\forall \sigma_k \in \Sigma_k$, $\forall a_k \in \{s_k, t_k\}$, $\forall a_{k'} \in \{s_{k'}, t_{k'}\}$, $\forall \tilde{\sigma} \in \tilde{\Sigma}_N$ such that $\forall k \in \{i, j\}$, $\tilde{\sigma}_k \notin \{s_k, t_k\} \cup \Sigma_k$, we have:

(i) $(\sigma_k, \tilde{\sigma}_{N\setminus\{k\}}) \tilde{\sim}_\ell (a_k, \tilde{\sigma}_{N\setminus\{k\}}) \tilde{\sim}_\ell (a_{k'}, \tilde{\sigma}_{N\setminus\{k'\}}) \tilde{\succ}_\ell \tilde{\sigma}$,

(ii) $(a_k, \sigma^*_{N\setminus\{k\}}) \tilde{\sim}_\ell (a_{k'}, \sigma^*_{N\setminus\{k'\}}) \tilde{\succ}_\ell (a_k, \tilde{\sigma}_{N\setminus\{k\}})$.

Rule 2. $\forall \ell \in N$, $\forall k \in \{i, j\}$, $\forall a_k \in \{s_k, t_k\}$, $\forall \sigma \in \Sigma_N$, we have $\sigma \tilde{\succ}_\ell (a_k, \sigma^*_{N\setminus\{k\}})$.

Rule 3. $\forall \ell \in N$, $\forall \sigma, \sigma' \in \Sigma_N$, $\sigma \succeq_\ell \sigma' \Leftrightarrow \sigma \tilde{\succeq}_\ell \sigma'$.

Rule 4. $\forall \ell \in N$, $\forall k \in \{i, j\}$, $\forall a_k \in \{s_k, t_k\}$, $\forall \sigma \in \Sigma_N$ such that $\sigma_{N\setminus\{k\}} \neq \sigma^*_{N\setminus\{k\}}$, we have $(a_k, \sigma_{N\setminus\{k\}}) \tilde{\succ}_\ell \sigma$.

Rule 5. $\forall \ell \in N$, $\forall k \in \{i, j\}$, $\forall a_k \in \{s_k, t_k\}$, $\forall a_i \in \{s_i, t_i\}$, $\forall a_j \in \{s_j, t_j\}$, $\forall \sigma \in \Sigma_N$ we have $(a_i, a_j, \sigma_{N\setminus\{i,j\}}) \tilde{\succ}_\ell (a_k, \sigma_{N\setminus\{k\}})$.

Rule 6. $\forall \ell \in N \setminus \{i, j\}$, $\forall \sigma \in \Sigma_N$, $\forall k \in \{i, j\}$, $\forall a_k \in \{s_k, t_k\}$, we have:

(i) $(s_i, s_j, \sigma_{N\setminus\{ij\}}) \tilde{\sim}_i (t_i, t_j, \sigma_{N\setminus\{ij\}}) \tilde{\succ}_i (s_i, t_j, \sigma_{N\setminus\{ij\}}) \tilde{\sim}_i (t_i, s_j, \sigma_{N\setminus\{ij\}})$,

(ii) $(s_i, t_j, \sigma_{N\setminus\{ij\}}) \tilde{\sim}_j (t_i, s_j, \sigma_{N\setminus\{ij\}}) \tilde{\succ}_j (s_i, s_j, \sigma_{N\setminus\{ij\}}) \tilde{\sim}_j (t_i, t_j, \sigma_{N\setminus\{ij\}})$,

(iii) $(s_i, t_j, \sigma_{N\setminus\{ij\}}) \tilde{\sim}_\ell (t_i, s_j, \sigma_{N\setminus\{ij\}}) \tilde{\sim}_\ell (s_i, s_j, \sigma_{N\setminus\{ij\}}) \tilde{\sim}_\ell (t_i, t_j, \sigma_{N\setminus\{ij\}})$.

Rule 7. $\forall \ell \in N$, $\tilde{\succeq}_\ell$ is transitive.

Observe that *Rules* 1 through 7 rank strategy profiles in increasing order according to players' preferences. They ensure that for all $\ell \in N$, $\tilde{\succeq}_\ell$ is a weak order of $\tilde{\Sigma}_N$. The proof that $(N, \Sigma_N, E^{NE}, \succeq)$ is an e-form subgame of $(N, \tilde{\Sigma}_N, \tilde{E}^{NE}, \tilde{\succeq})$ is an immediate consequence of *Rule* 3. Observe that *Rules* 2 and 4 ensure that individual deviations

of i and j from σ^* to $\tilde{\Sigma}_N \setminus \Sigma_N$ decrease the situation of all players, while individual deviations of i or j from $\Sigma_N \setminus \{\sigma^*\}$ to $\tilde{\Sigma}_N \setminus \Sigma_N$ improve the situation of all players. We want to prove that $C(\tilde{\Gamma}) = \{\sigma^*\}$. Let $\bar{\sigma} \in \tilde{\Sigma}_N \setminus \{\sigma^*\}$. Several cases arise:

(i) If $\bar{\sigma} \in \Sigma_N \setminus \{\sigma^*\}$, then two subcases can occur:

(a) If $\bar{\sigma}_{N\setminus\{i\}} \neq \sigma^*_{N\setminus\{i\}}$, then by *Rule* 4, we have $(s_i, \bar{\sigma}_{N\setminus\{i\}}) \tilde{\succ}_i \bar{\sigma}$;

(b) If $\bar{\sigma}_{N\setminus\{i\}} = \sigma^*_{N\setminus\{i\}}$, then $\bar{\sigma}_{N\setminus\{j\}} \neq \sigma^*_{N\setminus\{j\}}$ because $\bar{\sigma} \neq \sigma^*$. Hence, it follows from *Rule* 4 that $(s_j, \bar{\sigma}_{N\setminus\{j\}}) \tilde{\succ}_j \bar{\sigma}$.

In both subcases, we conclude that $\bar{\sigma}$ does not belong to $C(\tilde{\Gamma})$.

(ii) If $\bar{\sigma} \in \tilde{\Sigma}_N \setminus \Sigma_N$, then two subcases can occur:

(a) If there does not exist $k \in \{i, j\}$ such that $\bar{\sigma}_k \in \{s_k, t_k\} \cup \Sigma_k$, then by *Rule* 1, we have $(s_i, \bar{\sigma}_{N\setminus\{i\}}) \tilde{\succ}_i \bar{\sigma}$.

(b) If there exists $k \in \{i, j\}$ such that $\bar{\sigma}_k \in \{s_k, t_k\} \cup \Sigma_k$, then three cases arise. In each of these cases, we denote by k' the unique element in $\{i, j\} \setminus \{k\}$.

(b_1) If $\bar{\sigma}_k \in \{s_k, t_k\}$ and $\bar{\sigma}_{k'} \in \{s_{k'}, t_{k'}\}$, then by (ii) and (iii) of *Rule* 6, there exists an improving deviation of i or j.

(b_2) If $\bar{\sigma}_k \in \{s_k, t_k\}$ and $\bar{\sigma}_{k'} \in \Sigma_{k'}$, then by *Rule* 5, there exists an improving deviation of k' (the case where $\bar{\sigma}_k \in \Sigma_k$ and $\bar{\sigma}_{k'} \in \{s_{k'}, t_{k'}\}$ is similar).

(b_3) If $\bar{\sigma}_k \in \{s_k, t_k\} \cup \Sigma_k$ and $\bar{\sigma}_{k'} \notin \{s_{k'}, t_{k'}\} \cup \Sigma_{k'}$, then for all $\sigma_{k'} \in \Sigma_{k'}$, $(\sigma_{k'}, \bar{\sigma}_{N\setminus\{k'\}})$ is an improving deviation of k' from $\bar{\sigma}$. On the one hand, if $(\sigma_{k'}, \sigma_{N\setminus\{k,k'\}}) = \sigma^*_{N\setminus\{k\}}$, then it follows from (ii) of *Rule* 1 that $(\sigma_{k'}, \bar{\sigma}_{N\setminus\{k'\}})$ is an improving deviation of k' from $\bar{\sigma}$. On the other hand, if $(\sigma_{k'}, \sigma_{N\setminus\{k,k'\}}) \neq \sigma^*_{N\setminus\{k\}}$, then it follows from *Rules* 1, 4 and 7 that $(\sigma_{k'}, \bar{\sigma}_{N\setminus\{k'\}})$ is an improving deviation of k' from $\bar{\sigma}$.

In all cases, we conclude that $\bar{\sigma}$ does not belong to $C(\tilde{\Gamma})$.

It remains to prove that σ^* belongs to $C(\tilde{\Gamma})$. First, since σ^* belongs to $C(\Gamma)$, it follows from *Rule* 3 that there is no possible deviation from σ^* to Σ_N. Second, it follows from *Rule* 2 that there does not exist improving deviation from σ^* to $\tilde{\Sigma}_N \setminus \Sigma_N$. Thus, we conclude that $C(\tilde{\Gamma}) = \{\sigma^*\}$. ■

PROOF of Proposition 5.6. By Propositions 4.1 and 5.5 and Lemma 4.3, it is sufficient to prove that $\mathcal{K}_C^{PS}$ satisfies the holdover property. Let $\Gamma = (N, A, E^{PS}, \succeq) \in \mathcal{K}_C^{PS}$ and take any $\overline{g} \in C(\Gamma)$. If $|N| = 1$, then there is nothing to prove. Assume that $|N| \geq 2$ and let $\overline{\Gamma} = (N', A', E'^{PS}, \succeq') \in \mathcal{K}^{PS}$ be an e-form game where $N' = N \cup \{\overline{i}\}$. Consider any player $\overline{j} \in N$ and define players' preferences in the following way:

Rule 1. For player $\overline{i} \in N'$, we assume that for all $g, g' \in A' \setminus A$, $g \succ'_{\overline{i}} g'$ iff $|g| > |g'|$; for all $g, g' \in A \setminus \{\overline{g}\}$, $g \sim'_{\overline{i}} g'$; for all $g \in A \setminus \{\overline{g}\}$ and all $g' \in A' \setminus (A \setminus \{\overline{g}\})$, $g' \succ'_{\overline{i}} g$; for

all $g \in A' \setminus (A \cup \{\overline{g} + \overline{ij}\})$, $g \succeq'_{\overline{i}} \overline{g}$ and $\overline{g} + \overline{ij} \sim'_{\overline{i}} \overline{g}$.

Rule 2. For player $\overline{j} \in N$, we assume that for all $g, g' \in A$, $g \succeq'_{\overline{j}} g'$ iff $g \succeq_{\overline{j}} g'$; for all $g \in A$ and all $g' \in A' \setminus A$, $g' \succ'_{\overline{j}} g$; and for all $g, g' \in A' \setminus A$, $g \succ'_{\overline{j}} g'$ iff $|g| < |g'|$.

Rule 3. For all players $k \in N \setminus \{\overline{j}\}$, we assume that for all $g, g' \in A$, $g \succeq'_k g'$ iff $g \succeq_k g'$; for all $g \in A$ and all $g' \in A' \setminus A$, $g \succ'_k g'$; for all $g, g' \in A' \setminus A$ such that $\overline{ij} \in g$ and $\overline{ij} \in g'$, $g \succ'_k g'$ iff $|g| > |g'|$; for all $g, g' \in A' \setminus A$ such that $\overline{ij} \notin g$ and $\overline{ij} \notin g'$, $g \succ'_k g'$ iff $|g| < |g'|$; for all $g, g' \in A' \setminus A$ such that $\overline{ij} \in g$ and $\overline{ij} \notin g'$, $g \succ'_k g'$.

First, observe that Γ is an e-form subgame of $\overline{\Gamma}$. It remains to prove that $C(\overline{\Gamma}) = \{\overline{g}\}$. By construction, it must be clear that $\overline{g}$ is pairwise stable in $\overline{\Gamma}$. Then assume that there exists a pairwise stable network $\tilde{g}$ in $\overline{\Gamma}$ such that $\tilde{g} \neq \overline{g}$. We distinguish two cases:

(i) First, if $\tilde{g} \in A$, then $\tilde{g} + \overline{ij} \succ'_l \tilde{g}$ for all $l \in \{\overline{i}, \overline{j}\}$. So, $\tilde{g}$ is not pairwise stable;

(ii) Second, if $\tilde{g} \in A' \setminus A$, we distinguish two subcases:

  (a) If $\overline{ij} \in \tilde{g}$ then two cases can occur:
    i. There exists $k \in N \setminus \{\overline{j}\}$ such that $k\overline{i} \notin \tilde{g}$. In this case, we have $\tilde{g} + k\overline{i} \succ'_l \tilde{g}$ for all $l \in \{k, \overline{i}\}$. So, $\tilde{g}$ is not pairwise stable;
    ii. For all $k \in N \setminus \{\overline{j}\}$, $k\overline{i} \in \tilde{g}$. In this case, we have $\tilde{g} - \overline{ij} \succ'_{\overline{j}} \tilde{g}$. So, $\tilde{g}$ is not pairwise stable.

  (b) If $\overline{ij} \notin \tilde{g}$, then there exists $k \in N \setminus \{\overline{j}\}$ such that $k\overline{i} \in \tilde{g}$ since $\tilde{g} \in A' \setminus A$. In this case, we have $\tilde{g} - k\overline{i} \succ'_k \tilde{g}$. So, $\tilde{g}$ is not pairwise stable, a contradiction.

Thus, we conclude that $C(\overline{\Gamma}) = \{\overline{g}\}$. ■

PROOF of Proposition 6.1. Let φ be a solution on $\mathcal{K} \subseteq \mathcal{K}^{NE}$. Let $\mathcal{K}_0 \subseteq \mathcal{K}$ be a set closed under reduction. Assume that φ satisfies **OPR** and **WCONS** on $\mathcal{K}_0$. Let σ be an element of $\varphi(\Gamma)$. Assume that there exists $i \in N$ and $\sigma'_i \in \Sigma_i$ such that for all $\sigma'' \neq (\sigma'_i, \sigma_{-i})$, $(\sigma'_i, \sigma_{-i}) \succ_i \sigma''$. Since $\sigma \in \varphi(\Gamma)$ and φ satisfies **WCONS**, we have $\sigma_i \in \varphi(\Gamma^{\{i\},\sigma})$. Since φ satisfies **OPR**, we have $\{\sigma'_i\} = \varphi(\Gamma^{\{i\},\sigma})$. Hence, we obtain $\sigma_i = \sigma'_i$, which means that φ satisfies **WIR**. ■

PROOF of Proposition 6.2. Let $\mathcal{K} \subseteq \mathcal{K}^{NE}$ be a class of e-form games, which is both extendable and closed under reduction, and φ be a solution on $\mathcal{K}$. We omit the proof that the Nash equilibrium satisfies the axioms. First, assume that φ satisfies **NE**, **OPR**, **IISt**, and **WCONS**. Since φ satisfies **OPR** and **WCONS**, then by a similar argument to Peleg and Tijs (1996)[Proposition 2.8], we obtain that φ is a subsolution of the Nash equilibrium. We conclude with Proposition 4.3 that φ is equal to the Nash equilibrium. Second, assume that φ satisfies **NE**, **OPR**, **IISt**, and **WDUM**. Since φ satisfies **WDUM** and **IISt**, then by Ray (2000)[Proposition 2], we obtain that φ satisfies **WCONS** and conclude with the previous argument that φ is equal to the

Nash equilibrium. ■

PROOF of Lemma 6.3. Let $\mathcal{K}$ be a class satisfying the ancestor property and φ be a proper solution of the core, defined on $\mathcal{K}$ and which satisfies **NE** and **CONS**. Since φ is a proper solution of the core, there exists $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ and $a^* \in C(\Gamma)$ such that $a^* \notin \varphi(\Gamma)$. By the ancestor property, there exists $\Gamma' = (N', A'_{N'}, E', \succeq') \in \mathcal{K}$, with $N' \supseteq N$, such that Γ is the reduced e-form game of Γ' with respect to a^* and N, and for all $a' \in C(\Gamma'), a'_N = a^*$. Since φ is a proper subsolution of the core satisfying **NE**, we deduce that there exists $a' \in \varphi(\Gamma') \subseteq C(\Gamma')$. Since φ satisfies **CONS**, it follows that $a'_N = a^* \in \varphi(\Gamma)$, which is a contradiction. ■

PROOF of Theorem 6.4. Let $\mathcal{K}$ be a class of e-form games satisfying the ancestor property. From Proposition 4.1 and Lemma 6.3, it suffices to prove that the core defined on $\mathcal{K}$ satisfies **NE**, **CU**, **MM**, and **CONS** on $\mathcal{K}_C$. By Theorem 4.4, it suffices to prove that the core satisfies **CONS** on $\mathcal{K}_C$. Let $\Gamma = (N, A, E, \succeq) \in \mathcal{K}_C$ and $\bar{a} \in C(\Gamma)$. Let $S \subsetneq N$ be a proper subcoalition of N and $\Gamma^{S,\bar{a}} = (S, A_S, E^{S,\bar{a}}, \succeq') \in \mathcal{K}$ be the reduced e-form game of Γ with respect to $\bar{a}$ and S. Assume for the sake of contradiction that $\bar{a}_S \notin C(\Gamma^{S,\bar{a}})$. Then, there exists an objection (T, B_S), with $T \in \mathcal{P}_0(S)$ and $B_S \in E^{S,\bar{a}}_{\bar{a}_S}(T)$, against $\bar{a}_S$ in $\Gamma^{S,\bar{a}}$. It follows that there exists $B \in E_{\bar{a}}(T)$ with $B = B_S \times \{\bar{a}_{N\setminus S}\}$, and for all $b \in B$ and all $i \in T$, $b \succ_i \bar{a}$, contradicting $\bar{a} \in C(\Gamma)$.■

PROOF of Proposition 6.7. To prove that any solution φ satisfying **SOUL** and **MM** on $\mathcal{K}^{SM}$ is a subsolution of the stable matchings, it suffices to adapt the proof of Proposition 4.1. The proof that the stable matchings is a subsolution of any solution φ that satisfies **CONS** and **NE** on $\mathcal{K}^{SM}$ is a direct consequence of Sasaki and Toda (1992)[Lemma 1, p. 222] and Lemma 6.3. ■

Independence of the axioms

Now, we comment on the logical independence of the axioms in Theorem 4.4. In the following examples, we consider any class $\mathcal{K}$ of e-form games satisfying the holdover property.

Example 6.8. Consider the solution φ_1 on $\mathcal{K}$ such that $\varphi_1(\Gamma) = \emptyset$ for all $\Gamma \in \mathcal{K}$. Then φ_1 violates **NE** but vacuously satisfies **CU**, **MM**, and **IIS** on $\mathcal{K}_C$.

Example 6.9. Consider the solution φ_2 on $\mathcal{K}$ such that $\varphi_2(\Gamma) = A$ for all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$. Then φ_2 satisfies **NE**, **MM**, and **IIS** but not **CU** on $\mathcal{K}_C$.

Example 6.10. Let $\geq_{\mathcal{A}}$ be a linear order over $\mathcal{A}$, $\overline{n} \in \mathbb{N}$, and $\mathcal{K}$ be a class of e-form games such that $|C(\Gamma)| \geq 2$ for some $\Gamma \in \mathcal{K}$. Consider the solution φ_3 on $\mathcal{K}$ such that $\varphi_3(\Gamma) = \emptyset$ for all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}\setminus\mathcal{K}_C$ and:

$$\varphi_3(\Gamma) = \begin{cases} \{a\} \subseteq C(\Gamma) : \forall b \in C(\Gamma),\ a \geq_{\mathcal{A}} b \text{ if } |N| < \overline{n}; \\ \\ C(\Gamma) \qquad\qquad\qquad\qquad\qquad\qquad \text{if } |N| \geq \overline{n}; \end{cases}$$

for all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}_C$.
Then φ_3 satisfies **NE** on $\mathcal{K}_C$. Take any $\Gamma \in \mathcal{K}_C$ and any $a \in \varphi_3(\Gamma)$ and assume that there exists $B \in E_a(S)$ for some $S \in \mathcal{P}_0(N)$ such that B is unanimously prefered by S. Assume for the sake of contradiction that $a \notin B$. Hence (S, B) is an objection against a, a contradiction with $\varphi_3(\Gamma) \subseteq C(\Gamma)$. We conclude that φ_3 satisfies **CU** on $\mathcal{K}_C$. To see that φ_3 satisfies **MM** as well, consider any $\Gamma = (N, A, E, \succeq), \Gamma' = (N, A, E, \succeq') \in \mathcal{K}_C$, and any $a \in \varphi_3(\Gamma)$ such that $\succeq'$ is a Maskin monotonic transformation of $\succeq$ at a. We distinguish two cases:

(i) if $|N| < \overline{n}$, then $\varphi_3(\Gamma) = \varphi_3(\Gamma') = \{a\}$;

(ii) if $|N| \geq \overline{n}$, then $a \in \varphi_3(\Gamma) = C(\Gamma)$.

Since the core satisfies **MM**, we conclude that $a \in C(\Gamma') = \varphi_3(\Gamma')$. In both cases, φ_3 satisfies **MM**. It follows from Theorem 4.4 that φ_3 violates **IIS**.

We need to introduce a new notation which will be useful for the definition of the next solution. For $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ and $a \in A$, let $\mathcal{B}_a(\Gamma)$ be the collection of sets $B \in E_a(S)$ that are unanimously preferred by S for some $S \in \mathcal{P}_0(N)$.

Example 6.11. Consider the solution φ_4 on $\mathcal{K}$ such that:

$$\varphi_4(\Gamma) = \begin{cases} C(\Gamma) & \text{if } \mathcal{B}_a(\Gamma) \neq \emptyset \text{ for some } a \in A; \\ \\ A & \text{otherwise.} \end{cases}$$

Then φ_4 satisfies **NE** on $\mathcal{K}_C$. Take any $\Gamma = (N, A, E, \succeq) \in \mathcal{K}_C$ and any $a \in \varphi_4(\Gamma)$ and assume that there exists $B \in E_a(S)$ for some $S \in \mathcal{P}_0(N)$ such that B is unanimously prefered by S. Assume for the sake of contradiction that $a \notin B$. Hence (S, B) is an objection against a, and so $a \notin C(\Gamma)$. However, it follows from $\mathcal{B}_a(\Gamma) \neq \emptyset$ that $\varphi_4(\Gamma) = C(\Gamma)$, a contradiction. We conclude that φ_4 satisfies **CU** on $\mathcal{K}_C$. To see that φ_4 complies with **IIS** as well, take any $\Gamma \in \mathcal{K}_C$ and any e-form subgame $\Gamma' = (N', A', E', \succeq') \in \mathcal{K}_C$, where $N' \in \mathcal{P}_0(N)$ and $A' \in \mathcal{P}_0(A)$ such that $\varphi_4(\Gamma) \subseteq A'$. We distinguish two cases:

(i) if $\mathcal{B}_a(\Gamma) = \emptyset$ for all $a \in A$, then by the definition of Γ', it holds that $\mathcal{B}_a(\Gamma') = \emptyset$ for all $a \in A'$. Hence we conclude that $\varphi_4(\Gamma) = \varphi_4(\Gamma') = A$ and $A = A'$;

(ii) if $\mathcal{B}_a(\Gamma) \neq \emptyset$ for some $a \in A$, then we distinguish two subcases:

  (a) if $\mathcal{B}_a(\Gamma') = \emptyset$ for all $a \in A$, then $\varphi_4(\Gamma') = A' \supseteq \varphi_4(\Gamma)$;
  (b) if $\mathcal{B}_a(\Gamma') \neq \emptyset$ for some $a \in A$, then $\varphi_4(\Gamma') = C(\Gamma') \supseteq C(\Gamma) = \varphi_4(\Gamma)$.

In all cases, $\varphi_4(\Gamma) \subseteq \varphi_4(\Gamma')$, which proves that φ_4 satisfies **IIS**. It follows from Theorem 4.4 that φ_4 violates **MM**.

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