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Dongshuang Hou
Aymeric Lardon
Panfei Sun
Theo Driessen

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Compromise for the per capita complaint: an optimization characterization of two equalitarian values

DONGSHUANG HOU * AYMERIC LARDON†
PANFEI SUN ‡ THEO DRIESSEN §

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Abstract

The main purpose of this article is to introduce two new values for transferable utility (TU) games: the upper and lower optimal complaint values. These are based on two kinds of per capita complaint criteria and each involve a lower and upper bound of the core (Shapley, 1955 [13]). In the spirit of the nucleolus (Schmeidler, 1969 [12]), these two values are obtained by lexicographically minimizing a maximal complaint vector associated with each of the per capita complaint criterion. Interestingly, the upper and lower optimal complaint values respectively coincide with the Equal Allocation of Non-Separable Contributions (Moulin, 1985 [10]) and the Center-of-Gravity of Imputation Set Value (Driessen and Funaki, 1991 [4]) for a large class of TU-games. Moreover, a characterization of these two values is achieved by invoking the equal upper and lower maximal per capita complaint properties together with efficiency.

Keywords: Cooperative game, optimal complaint values, equalitarian values, equal maximal per capita complaint properties
JEL Classification System: C71
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*Dongshuang Hou, Department of Applied Mathematics, Northwestern Polytechnical University. E-mail: dshhou@126.com The first author acknowledges financial support by National Science Foundation of China (NSFC) through grant No.71601156, 71571140, 71571143.

†Aymeric Lardon, Université Côte d'Azur, CNRS, GREDEG, France. This work has been supported through the UCA$^{\text{JEDI}}$ Investments in the Future project managed by the French National Research Agency (ANR) under grant ANR-15-IDEX-01.

‡Panfei Sun, Department of Applied Mathematics, Northwestern Polytechnical University.

§Theo Driessen, Department of Applied Mathematics, University of Twente, The Netherlands.

1 Introduction

Cooperative game theory is a mathematical framework to model and analyze allocation problems within a group of players establishing a contract on a joint plan of activities. A cooperative game with transferable utility (henceforth a TU-game) on a given set of players is a function which specifies, for each coalition of players, the amount its members can achieve without the help of any other player. A solution associates with each TU-game a non-empty set of payoff vectors that generally represents the possible allocations of the worth created when all players cooperate within the grand coalition. A value is a (single-valued) solution which assigns a single payoff to each player in the TU-game. The payoff received by a player can then be interpreted as his or her gains for participating in the TU-game. The core, one of the most fundamental (set-valued) solution concepts for TU-games, was proposed by Shapley (1955) [13]. Roughly speaking, no coalition has an incentive to deviate from the grand coalition and guarantee its members a larger payoff on a core allocation.

Once an allocation is made in a TU-game, players may be dissatisfied about their received payoffs and express an amount of complaint within a coalition. Davis and Maschler (1965) [3] introduced the complaint of a coalition based on the excess criterion to determine the kernel of TU-games. The excess of a coalition manifests its members' potential to generate more utility if they decide to contest an allocation. In the same vein, Grotte (1970) [5] proposed the per capita excess, which, for each coalition, measures the dissatisfaction per player about a payoff distribution. In this article, we propose alternative measures of players' dissatisfaction about an allocation in terms of per capita complaint involving a lower and upper bound of the core. For a TU-game with a non-empty core, a player's marginal contributions to the grand coalition can be seen as his or her ideal payoff since this corresponds to an upper bound of the core. Thus, when the sum of players' marginal contributions to the grand coalition cannot be covered by the worth of the grand coalition, some of them don't receive their expected payoffs and express an amount of complaint about a core allocation. Alternatively, coalition members may be satisfied if the other players only receive their individual amounts which constitute a lower bound of the core. Thereby, when the sum of players' individual worths can be covered by the worth of the grand coalition, players may complain about a core allocation since some other players obtain payoffs strictly greater than their individual worths. Based on the idea that the core is bounded above by players' marginal contributions to the grand coalition, for each coalition, we use the average gap between the sum of its members' marginal contributions to the grand coalition and the sum of their payoffs to determine the upper per capita complaint about an allocation. On the other hand, since the core is bounded below by players' individual worths, for each coalition, we compute the average gap between the sum of its non-members' payoffs and the sum of their individual worths to define the lower per capita complaint about a payoff distribution.

From these two new measures of players' dissatisfaction, we define the upper and lower optimal complaint values, which, in the spirit of the nucleolus (Schmeidler, 1969 [12]), are obtained by lexicographically minimizing a maximal complaint vector associated with each of the two per capita complaint criteria described above. The most salient properties of these two values stems from the fact that, for a large class of TU-games, they respectively coincide with two equalitarian values, namely the Equal Allocation of Non-Separable Contributions (Moulin, 1985 [10]) and the Center-of-Gravity of Imputation Set Value (Driessen and Funaki, 1991 [4]). More precisely, when the sum of players' marginal contributions to the grand coalition cannot be covered by the worth of the grand coalition, the upper optimal complaint value is bounded above by such contributions. Moreover, when the sum of players' individual worths can be covered by the worth of the grand coalition, the lower optimal complaint value is bounded below by the individual worths.

The Equal Allocation of Non-Separable Contributions and the Center-of-Gravity of Imputation Set Value have received considerable attention over the past few years, including several works focused on the axiomatic characterization of these two values invoking consistency principles together with other standard axioms for TU-games (Hwang, 2006 [7]; Xu *et al.*, 2013 [16], 2015 [15]).[1] Another interesting approach adopted by Béal *et al.* (2016) [1] uses the axiom of balanced collective contributions, inspired by the axiom of balanced contributions proposed by Myerson (1980) [11], and the classical axiom of efficiency to characterize the Equal Allocation of Non-Separable Contributions. In this article, we focus on the axiomatic characterization of these two equalitarian values. Several solution concepts such as the (per capita) nucleolus and the kernel have been characterized on the basis of the optimization approach (see, for example, Huijink *et al.*, 2015 [6]). Drawing on the upper and lower per capita complaint criteria, we propose two new axioms: the equal upper and lower maximal per capita complaint properties. These are inspired by the idea of the (pre-) kernel (Maschler and Peleg, 1966 [8], Maschler *et al.*, 1972 [9]), which, together with efficiency, uniquely determine the Equal Allocation of Non-Separable Contributions and the Center-of-Gravity of Imputation Set Value.

The article is organized as follows. Section 2 provides basic definitions and notations. The two optimal complaint values are introduced and studied in Section 3, with Section 4 devoted to their axiomatic characterizations. Finally, Section 5 concludes.

2 Definitions and notations

Let $N = \{1, 2, \ldots, n\}$ be a fixed and finite set of n players. An element $i \in N$ is a player, while a non-empty subset $S \subseteq N$ is a coalition, whose cardinality will be denoted by $|S|$. A cooperative game with transferable utility, or simply a **TU-game**, on the set N of players is a real-valued function v defined on the set 2^N of all subsets of N such that $v(\emptyset) = 0$. For each coalition $S \subseteq N$, $v(S)$ describes the value that coalition S can achieve when all its members cooperate. The set of all TU-games on N is denoted by Γ^N. An **allocation** among the players is given by a n-tuple vector $x = (x_1, x_2, \cdots, x_n)$, called a payoff vector, whose components are real numbers. For notational convenience, we will write $\Sigma_{i \in S} x_i$ as $x(S)$.

A **solution** on Γ^N is a function φ, which associates with each TU-game $v \in \Gamma^N$, a subset

[1] We refer to Béal *et al.* (2016) [2] for unified proofs of these results.

$\varphi(v) \subseteq R^n$ of payoff vectors. If φ assigns a unique payoff vector to each TU-game $v \in \Gamma^N$, then φ is called a **value**. For a TU-game $v \in \Gamma^N$, the **pre-imputation set** $I^*(v)$ is defined as:

$$I^*(v) = \{x \in R^n \mid x(N) = v(N)\}.$$

Each element of $I^*(v)$ is a pre-imputation, whose the worth of the grand coalition is fully distributed among the players. The additional individual rationality condition permits to characterize the **imputation set** $I(v)$ defined as:

$$I(v) = \{x \in I^*(v) \mid \forall i \in N,\ x_i \geq v(\{i\})\}.$$

Each element of $I(v)$ is an imputation, whose players receive payoffs larger than or equal to their individual worths. It is worth noting that $I(v) \neq \emptyset$ if and only if $\sum_{j \in N} v(\{j\}) \leq v(N)$. The **core** $C(v)$ (Shapley, 1955 [13]) is defined through group rationality and efficiency conditions as follows:

$$C(v) = \{x \in I^*(v) \mid \forall S \subseteq N,\ x(S) \geq v(S)\}.$$

The set of TU-games with a non-empty core is denoted by Γ_0^N.
For a TU-game $v \in \Gamma^N$ and a payoff vector $x \in R^n$, the **excess** of a coalition with respect to x (Davis and Maschler, 1965 [3]) is defined as:

$$\forall S \subseteq N,\ e^v(S) = v(S) - x(S).$$

The excess of a coalition can be viewed as its complaint about a payoff distribution, i.e. the higher the excess is, the more dissatisfied coalition members feel. The excess criterion serves as the basis of several solution concepts for TU-games, such as the core (Shapley, 1955 [13]), the nucleolus (Schemeidler, 1969 [12]), and the kernel (Davis and Maschler, 1965 [3]). For example, the core results in non-positive excesses.
In the next section, we will propose two new per capita complaint criteria, which will turn out to be closely related to well-known equalitarian values for TU-games introduced now.
The **Equal Allocation of Non-Separable Contributions** (Moulin, 1985 [10]), denoted by EANSC(v), is the value defined as:

$$\forall i \in N,\ \mathrm{EANSC}_i(v) = b_i^v + \frac{1}{n}\Big(v(N) - \sum_{j \in N} b_j^v\Big),$$

where,

$$\forall i \in N,\ b_i^v = v(N) - v(N \setminus \{i\}).$$

The EANSC value is characterized by the fact that after each player has obtained his or her marginal contribution to the grand coalition, the rest of non-separable contributions is equally distributed among its players.
The **Center-of-Gravity of Imputation Set Value** (Driessen and Funaki, 1991 [4]),

denoted by CIS(v), is the dual of the EANSC value, where the dual v^D of a TU-game $v \in \Gamma^N$ is given by:

$$\forall S \subseteq N,\, S \neq \emptyset,\, v^D(S) = v(N) - v(S).$$

Therefore, the CIS value is defined as:

$$\begin{aligned}\forall i \in N,\, \mathrm{CIS}_i(v) &= \mathrm{EANSC}_i(v^D),\\ &= v(\{i\}) + \frac{1}{n}\Big(v(N) - \sum_{j \in N} v(\{j\})\Big).\end{aligned}$$

The CIS value gives to each player his or her individual worth and then equally distributes the remaining worth of the grand coalition among its players.

3 The optimal complaint values

In this section, we define two optimal complaint values, each based on a new per capita complaint criterion described in the next two subsections. Similar to the nucleolus (Schmeidler, 1969 [12]), these values are obtained by lexicographically minimizing a maximal complaint vector associated with each of the two per capita complaint criteria. To this end, we introduce the **lexicographical ordering** $\leq_L$ of R^{2^n-1}. For all $x, y \in R^{2^n-1}$, we say that x is lexicographically less than or equal to y, denoted by $x \leq_L y$, if $x = y$ or if there exists $t \in \{1, 2, \ldots, 2^n - 1\}$ such that $x_l = y_l$ for all $l \in \{1, 2, \ldots, t-1\}$ and $x_t < y_t$. Moreover, for a payoff vector $x \in R^n$, we denote by $\theta(x) \in R^{2^n-1}$ the associated **complaint vector** whose components are the complaints of the coalitions with respect to x arranged in non-increasing order, i.e. $\theta_j(x) \leq \theta_i(x)$ for all $i, j \in \{1, \ldots, 2^n - 1\}$ such that $i \leq j$. For example, the nucleolus consists in lexicographically minimizing the maximal complaint vector $\theta(x)$ where the complaints of the coalitions are computed by means of the excess criterion defined above.

3.1 The upper optimal complaint value

For a TU-game $v \in \Gamma^N$, players' marginal contributions to the grand coalition constitute an upper bound of the core since for each $x \in C(v)$, $x_i \leq b_i^v$ for all $i \in N$. From the upper bound b^v, the **upper per capita complaint** of a coalition is defined as:

$$\forall S \subseteq N,\, S \neq \emptyset,\, \overline{e}(S, x, b^v) = \frac{b^v(S) - x(S)}{|S|}, \tag{3.1}$$

where $b^v(S) = \sum_{i \in S} b_i^v$. Thereby, the coalition members hope to obtain at least their marginal contributions to the grand coalition and express a positive amount of per capita complaint if their payoffs are below these expectations.

The **upper optimal complaint value** corresponds to the unique pre-imputation $x^{UOC} \in$

$I^*(v)$ and whose complaint vector $\theta(x^{UOC})$ derived from (3.1) possesses the smallest lexicographic order $\leq_L$, i.e.

$$\forall y \in I^*(v),\ \theta(x^{UOC}) \leq_L \theta(y).$$

Lemma 3.1. Given a TU-game $v \in \Gamma^N$ and a payoff vector $x \in R^n$, let $T_1 = \{k \in N \mid \forall l \in N,\ b_k^v - x_k \geq b_l^v - x_l\}$. If $p \in T_1$, then the largest per capita complaint with respect to x derived from (3.1) is given by $\theta_1(x) = b_p^v - x_p$.

Proof. Consider any player $p \in T_1$. By the definition of T_1, for any coalition $S \subseteq N$:

$$\begin{aligned} b_p^v - x_p &\geq \frac{\sum_{k\in S}(b_k^v - x_k)}{|S|}, \\ &= \frac{b^v(S) - x(S)}{|S|}. \end{aligned}$$

Thus, we conclude from (3.1) that $\theta_1(x) = b_p^v - x_p$. □

Proposition 3.2. Given a TU-game $v \in \Gamma_0^N$, the upper optimal complaint value $x^{UOC} \in I^*(v)$ is bounded above by b^v, i.e. $x_i^{UOC} \leq b_i^v$ for all $i \in N$.

Proof. Assume, for the sake of contradiction, that $x_i^{UOC} \leq b_i^v$ does not hold for all $i \in N$. We distinguish two cases:
Case 1: Suppose that $x_i^{UOC} \geq b_i^v$ for all $i \in N$ and there exists at least one player $j \in N$ such that $x_j^{UOC} > b_j$. Then, it holds that:

$$x^{UOC}(N) > b^v(N) \geq v(N), \tag{3.2}$$

where the last inequality holds because $v \in \Gamma_0^N$. Therefore, $x^{UOC}(N) > v(N)$ is a contradiction for $x^{UOC} \in I^*(v)$.
Case 2: Suppose that there exist at least two players $s, t \in N$ such that $x_s^{UOC} < b_s^v$ and $x_t^{UOC} > b_t^v$. Let $T_1 = \{k \in N \mid \forall l \in N,\ b_k^v - x_k^{UOC} \geq b_l^v - x_l^{UOC}\}$ and $T_2 = \{k \in N \setminus T_1 \mid \forall l \in N \setminus T_1,\ b_k^v - x_k^{UOC} \geq b_l^v - x_l^{UOC}\}$. Here, T_1 and T_2 are manifestly non-empty sets such that $T_1 \cap T_2 = \emptyset$. We take any $p \in T_1$ and any $q \in T_2$ and define $\triangle = \min\{b_p^v - x_p^{UOC}, x_t^{UOC} - b_t^v\} > 0$. The new payoff vector $x^* \in R^n$ as follows:

$$x_k^* = \begin{cases} x_k^{UOC} & \text{if } k \neq t \text{ and } k \notin T_1; \\ x_k^{UOC} + \frac{\triangle}{|T_1|} & \text{if } k \in T_1; \\ x_k^{UOC} - \triangle & \text{if } k = t. \end{cases} \tag{3.3}$$

Next, we distinguish two subcases:
Subcase 1: Assume that $q \neq t$. It follows from (3.3) that:

$$
\begin{aligned}
b_t^v - x_t^* &= b_t^v - x_t^{UOC} + \triangle, \\
&= b_t^v - x_t^{UOC} + \min\{b_p^v - x_p^{UOC}, x_t^{UOC} - b_t^v\}, \\
&= \min\{b_t^v - x_t^{UOC} + b_p^v - x_p^{UOC}, 0\}, \\
&\leq 0,
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
b_p^v - x_p^* &= b_p^v - x_p^{UOC} - \frac{\triangle}{|T_1|}, \\
&= b_p^v - x_p^{UOC} - \frac{\min\{b_p^v - x_p^{UOC}, x_t^{UOC} - b_t^v\}}{|T_1|}, \\
&\geq 0.
\end{aligned} \tag{3.5}
$$

We deduce from (3.4) and (3.5) that:

$$
b_p^v - x_p^* \geq b_t^v - x_t^*. \tag{3.6}
$$

Moreover, for all $s \in N \setminus (T_1 \cup T_2 \cup \{t\})$, it follows from (3.3) that:

$$
\begin{aligned}
b_q^v - x_q^* &= b_q^v - x_q^{UOC}, \\
&> b_s^v - x_s^{UOC}, \\
&= b_s^v - x_s^*,
\end{aligned} \tag{3.7}
$$

where the inequality holds because $q \in T_2$ and $s \notin T_1 \cup T_2$. Therefore, by Lemma 3.1, (3.6), and (3.7):

$$
\begin{aligned}
\theta_1(x^*) &= \max\{b_k^v - x_k^*,\ k \in N\}, \\
&= \max\{b_p^v - x_p^*, b_q^v - x_q^*, b_t^v - x_t^*, b_s^v - x_s^*,\ s \in N \setminus (T_1 \cup T_2 \cup \{t\})\}, \\
&= \max\{b_p^v - x_p^*, b_q^v - x_q^*\}, \\
&= \max\{b_p^v - x_p^{UOC} - \frac{\triangle}{|T_1|}, b_q^v - x_q^{UOC}\}, \\
&< b_p^v - x_p^{UOC}, \\
&= \theta_1(x^{UOC}).
\end{aligned}
$$

Hence, we conclude that $\theta(x^{UOC}) >_L \theta(x^*)$, which is a contradiction.
Subcase 2: Assume that $q = t$. It follows from (3.3) that:

$$\begin{aligned}
b_q^v - x_q^* &= b_q^v - x_q^{UOC} + \triangle, \\
&= b_q^v - x_q^{UOC} + \min\{b_p^v - x_p^{UOC}, x_q^{UOC} - b_q^v\}, \\
&= \min\{b_q^v - x_q^{UOC} + b_p^v - x_p^{UOC}, 0\}, \\
&\leq 0. \qquad (3.8)
\end{aligned}$$

We deduce from (3.5) and (3.8) that:

$$b_p^v - x_p^* \geq b_q^v - x_q^*. \qquad (3.9)$$

Therefore, by Lemma 3.1 and (3.9):

$$\begin{aligned}
\theta_1(x^*) &= \max\{b_k^v - x_k^*,\ k \in N\}, \\
&= \max\{b_p^v - x_p^*, b_q^v - x_q^*, b_s^v - x_s^*,\ s \in N \setminus (T_1 \cup T_2)\}, \\
&= \max\{b_p^v - x_p^*, b_s^v - x_s^*,\ s \in N \setminus (T_1 \cup T_2)\}, \\
&= \max\{b_p^v - x_p^{UOC} - \frac{\triangle}{|T_1|}, b_s^v - x_s^{UOC},\ s \in N \setminus (T_1 \cup T_2)\}, \\
&< b_p^v - x_p^{UOC}, \\
&= \theta_1(x^{UOC}).
\end{aligned}$$

Hence, we conclude that $\theta(x^{UOC}) >_L \theta(x^*)$, which is a contradiction. □

The important second inequality in (3.2) should draw our attention to the set of TU-games where the worth $v(N)$ cannot cover the sum of the marginal contributions, i.e. $v(N) \leq b^v(N)$. We denote this set of TU-games by Γ_1^N, for which the conclusion in Proposition 3.2 still holds.

Theorem 3.3. Given a TU-game $v \in \Gamma_1^N$, the upper optimal complaint value $x^{UOC} \in I^*(v)$ is bounded above by b^v.

The proof is omitted since it is similar to that of Proposition 3.2. By Theorem 3.3, we can narrow the scope of the upper optimal complaint value from the pre-imputation set $I^*(v)$ to the set $K = \{x \in I^*(v) \mid \forall i \in N,\ x_i \leq b_i^v\}$. The following lemma compares the upper per capita complaints within a coalition for two given payoff vectors.

Lemma 3.4. Given a TU-game $v \in \Gamma_1^N$, a payoff vector $x \in K$, and two players $l, m \in N$ such that $b_l^v - x_l > b_m^v - x_m \geq 0$, let $\triangle = \frac{b_l^v - x_l - (b_m^v - x_m)}{2} > 0$ and the new payoff vector $x^* \in K$ be defined as follows:

$$x_k^* = \begin{cases} x_k & \text{if } k \notin \{l, m\}; \\ x_l + \triangle & \text{if } k = l; \\ x_m - \triangle & \text{if } k = m. \end{cases} \qquad (3.10)$$

Then, the following five statements hold:

(i) $\overline{e}(S,x^*,b^v)=\overline{e}(S,x,b^v)$ for any $S\subseteq N$ such that $l,m\notin S$;

(ii) $\overline{e}(S,x^*,b^v)=\overline{e}(S,x,b^v)$ for any $S\subseteq N$ such that $l,m\in S$;

(iii) $\overline{e}(S,x^*,b^v)<\overline{e}(S,x,b^v)$ for any $S\subseteq N$ such that $l\in S$ and $m\notin S$;

(iv) For any $S\subseteq N$ such that $l\notin S$ and $m\in S$, it is redundant to consider the relation between $\overline{e}(S,x,b^v)$ and $\overline{e}(S,x^*,b^v)$ when comparing $\theta(x)$ and $\theta(x^*)$ in the lexicographic order;

(v) $\theta(x)>_L\theta(x^*)$.

Proof. (i) By the definition of $x^*\in K$, it is immediate that for any coalition $S\subseteq N$ with $l,m\notin S$, we have $\overline{e}(S,x,b^v)=\overline{e}(S,x^*,b^v)$.
(ii) For any $S\subseteq N$ such that $l,m\in S$:

$$\begin{aligned}\overline{e}(S,x^*,b^v) &= \frac{b^v(S)-x^*(S)}{|S|},\\ &= \frac{b^v(S\backslash\{l,m\})-x^*(S\backslash\{l,m\})+(b^v_l-x^*_l)+(b^v_m-x^*_m)}{|S|},\\ &= \frac{b^v(S\backslash\{l,m\})-x(S\backslash\{l,m\})+(b^v_l-(x_l+\triangle))+(b^v_m-(x_m-\triangle))}{|S|},\\ &= \frac{b^v(S)-x(S)}{|S|},\\ &= \overline{e}(S,x,b^v).\end{aligned}$$

(iii) For any $S\subseteq N$ such that $l\in S$ and $m\notin S$, it holds that:

$$\begin{aligned}\overline{e}(S,x^*,b^v) &= \frac{b^v(S)-x^*(S)}{|S|},\\ &= \frac{b^v(S\backslash\{l\})-x^*(S\backslash\{l\})+(b^v_l-x^*_l)}{|S|},\\ &= \frac{b^v(S\backslash\{l\})-x(S\backslash\{l\})+(b^v_l-(x_l+\triangle))}{|S|},\\ &= \frac{b^v(S)-x(S)-\triangle}{|S|},\\ &= \overline{e}(S,x,b^v)-\frac{\triangle}{|S|},\\ &< \overline{e}(S,x,b^v),\end{aligned}$$

where the last inequality holds because $\triangle>0$.

(iv) For any $S \subseteq N$ such that $l \notin S$ and $m \in S$:

$$\begin{aligned}
\overline{e}(S,x,b^v) &= \frac{b^v(S) - x(S)}{|S|}, \\
&= \frac{b^v(S\backslash\{m\}) - x(S\backslash\{m\}) + (b^v_m - x_m)}{|S|}, \\
&< \frac{b^v(S\backslash\{m\}) - x(S\backslash\{m\}) + (b^v_l - x_l)}{|S|}, \\
&= \frac{b^v(S\cup\{l\}\backslash\{m\}) - x(S\cup\{l\}\backslash\{m\})}{|S|}, \\
&= \overline{e}(S\cup\{l\}\backslash\{m\}, x, b^v). \qquad (3.11)
\end{aligned}$$

Therefore, for any $S \subseteq N$ such that $l \notin S$ and $m \in S$, there always exists a coalition $S\cup\{l\}\backslash\{m\}$ for which $\overline{e}(S\cup\{l\}\backslash\{m\}, x, b^v) > \overline{e}(S,x,b^v)$. Moreover, it follows from $b^v_l - x^*_l = b^v_m - x^*_m$ that:

$$\begin{aligned}
\overline{e}(S,x^*,b^v) &= \frac{b^v(S) - x^*(S)}{|S|}, \\
&= \frac{b^v(S\backslash\{m\}) - x^*(S\backslash\{m\}) + (b^v_m - x^*_m)}{|S|}, \\
&= \frac{b^v(S\backslash\{m\}) - x^*(S\backslash\{m\}) + (b^v_l - x^*_l)}{|S|}, \\
&= \frac{b^v(S\cup\{l\}\backslash\{m\}) - x^*(S\cup\{l\}\backslash\{m\})}{|S|}, \\
&= \overline{e}(S\cup\{l\}\backslash\{m\}, x^*, b^v). \qquad (3.12)
\end{aligned}$$

Thus, by (3.11), (3.12), and the inequality $\overline{e}(S\cup\{l\}\backslash\{m\}, x^*, b^v) < \overline{e}(S\cup\{l\}\backslash\{m\}, x, b^v)$ which holds by (iii), it is redundant to consider the relation between $\overline{e}(S,x,b^v)$ and $\overline{e}(S,x^*,b^v)$ when comparing $\theta(x)$ and $\theta(x^*)$ in the lexicographic order.

(v) Statement (v) follows directly from (i), (ii), (iii), and (iv). □

We now show that the upper optimal value coincides with the Equal Allocation of Non-Separable Contributions on Γ^N_1.

Theorem 3.5. Given a TU-game $v \in \Gamma^N_1$ and the upper optimal complaint value $x^{UOC} \in I^*(v)$, it holds that:

(i) $b^v_i - x^{UOC}_i$ is a constant for all $i \in N$;

(ii) $x^{UOC} = \text{EANSC}(v)$.

Proof. (i) Assume, for the sake of contradiction, that $b_i^v - x_i^{UOC}$ is not a constant for all $i \in N$, i.e. there exist $l, m \in N$ such that $b_l^v - x_l^{UOC} \neq b_m^v - x_m^{UOC}$. Without loss of generality, we suppose that $b_l^v - x_l^{UOC} > b_m^v - x_m^{UOC}$. By Theorem 3.3 and statement (v) of Lemma 3.4, there exists a payoff vector $x^* \in K$ such that $\theta(x^{UOC}) >_L \theta(x^*)$, a contradiction.
(ii) It follows from point (i) and $x^{UOC} \in I^*(v)$ that $x_i^{UOC} = \mathrm{EANSC}_i(v)$ for all $i \in N$. $\square$

3.2 The lower optimal complaint value

For a TU-game $v \in \Gamma^N$, players' individual worths correspond to a lower bound of the core since for each $x \in C(v)$, $v(\{i\}) \leq x_i$ for all $i \in N$. From the lower bound $(v(\{i\}))_{i \in N}$, the **lower per capita complaint** of a coalition (including the empty set) is defined as:

$$\forall S \subsetneq N,\ \overline{e}(S, x, (v(\{i\}))_{i \in N}) = \frac{x(N \backslash S) - \sum_{j \in N \backslash S} v(\{j\})}{|N \backslash S|}. \tag{3.13}$$

Thus, the coalition members hope that other players get nothing more than their individual worths and express a positive amount of complaint if their payoffs are above these expectations.
The **lower optimal complaint value** corresponds to the unique pre-imputation $x^{LOC} \in I^*(v)$ and whose complaint vector $\theta(x^{LOC})$ derived from (3.13) possesses the smallest lexicographic order $\leq_L$, i.e.

$$\forall y \in I^*(v),\ \theta(x^{LOC}) \leq_L \theta(y).$$

Lemma 3.6. Given a TU-game $v \in \Gamma^N$ and a payoff vector $x \in R^n$, let $Q_1 = \{k \in N \mid \forall l \in N,\ x_k - v(\{k\}) \geq x_l - v(\{l\})\}$. If $p \in Q_1$, then the largest per capita complaint with respect to x derived from (3.13) is given by $\theta_1(x) = x_p - v(\{p\})$.

Proof. Consider any player $p \in Q_1$. By the definition of Q_1, for any coalition $S \subseteq N$:

$$\begin{aligned} x_p - v(\{p\}) &\geq \frac{\sum_{k \in S}(x_k - v(\{k\}))}{|S|}, \\ &= \frac{x(S) - \sum_{k \in S} v(\{k\})}{|S|}, \end{aligned}$$

or equivalently, for any $S \subsetneq N$,

$$x_p - v(\{p\}) \geq \frac{x(N \setminus S) - \sum_{k \in N \setminus S} v(\{k\})}{|N \setminus S|}.$$

Thus, we conclude from (3.13) that $\theta_1(x) = x_p - v(\{p\})$. $\square$

Proposition 3.7. Given a TU-game $v \in \Gamma_0^N$, the lower optimal complaint value $x^{LOC} \in I^*(v)$ is bounded below by $(v(\{i\}))_{i \in N}$, i.e. $x_i^{LOC} \geq v(\{i\})$ for all $i \in N$.

Proof. Assume, for the sake of contradiction, that $x_i^{LOC} \geq v(\{i\})$ does not hold for all $i \in N$. We distinguish two cases:
Case 1: Suppose that $x_i^{LOC} \leq v(\{i\})$ for all $i \in N$ and there exists at least one player $j \in N$ such that $x_j^{LOC} < v(\{j\})$. Then, it holds that:

$$x^{LOC}(N) < \sum_{i \in N} v(\{i\}) \leq v(N), \tag{3.14}$$

where the last inequality holds because $v \in \Gamma_0^N$. Therefore, $x^{LOC}(N) < v(N)$, which is a contradiction since $x^{LOC} \in I^*(v)$.
Case 2: Suppose that there are at least two players $s, t \in N$ such that $x_s^{LOC} < v(\{s\})$ and $x_t^{LOC} > v(\{t\})$. Let $Q_1 = \{k \in N \mid \forall l \in N,\ x_k^{LOC} - v(\{k\}) \geq x_l^{LOC} - v(\{l\})\}$ and $Q_2 = \{k \in N \setminus Q_1 \mid \forall l \in N \setminus Q_1,\ x_k^{LOC} - v(\{k\}) \geq x_l^{LOC} - v(\{l\})\}$. It is manifest that Q_1 and Q_2 are non-empty sets such that $Q_1 \cap Q_2 = \emptyset$. We consider any $p \in Q_1$ and any $q \in Q_2$ and denote $\triangle = \min\{x_p^{LOC} - v(\{p\}), v(\{s\}) - x_s^{LOC}\} > 0$. The new payoff vector $x^* \in R^n$ is constructed as follows:

$$x_k^* = \begin{cases} x_k^{LOC} & \text{if } k \neq s \text{ and } k \notin Q_1; \\ x_k^{LOC} - \frac{\triangle}{|Q_1|} & \text{if } k \in Q_1; \\ x_s^{LOC} + \triangle & \text{if } k = s. \end{cases} \tag{3.15}$$

Next, we distinguish two subcases:
Subcase 1: Assume that $q \neq s$. It follows from (3.15) that:

$$\begin{aligned} x_s^* - v(\{s\}) &= x_s^{LOC} + \triangle - v(\{s\}), \\ &= x_s^{LOC} - v(\{s\}) + \min\{x_p^{LOC} - v(\{p\}), v(\{s\}) - x_s^{LOC}\}, \\ &= \min\{x_s^{LOC} - v(\{s\}) + x_p^{LOC} - v(\{p\}), 0\}, \\ &\leq 0, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} x_p^* - v(\{p\}) &= x_p^{LOC} - v(\{p\}) - \frac{\triangle}{|Q_1|}, \\ &= x_p^{LOC} - v(\{p\}) - \frac{\min\{x_p^{LOC} - v(\{p\}), v(\{s\}) - x_s^{LOC}\}}{|Q_1|}, \\ &\geq 0. \end{aligned} \tag{3.17}$$

We deduce from (3.16) and (3.17) that:

$$x_p^* - v(\{p\}) \geq x_s^* - v(\{s\}). \tag{3.18}$$

Therefore, by Lemma 3.6, (3.15), and (3.18):

$$\begin{aligned}
\theta_1(x^*) &= \max\{x^*_k - v(\{k\}),\ k \in N\}, \\
&= \max\{x^*_p - v(\{p\}), x^*_q - v(\{q\})\}, \\
&= \max\{x^{LOC}_p - v(\{p\}) - \frac{\triangle}{|Q_1|}, x^{LOC}_q - v(\{q\})\}, \\
&< x^{LOC}_p - v(\{p\}), \\
&= \theta_1(x^{LOC}).
\end{aligned}$$

Hence, we conclude that $\theta(x^{LOC}) >_L \theta(x^*)$, a contradiction.
Subcase 2: Assume that $q = s$. It follows from (3.15) that:

$$\begin{aligned}
x^*_q - v(\{q\}) &= x^{LOC}_q + \triangle - v(\{q\}), \\
&= x^{LOC}_q - v(\{q\}) + \min\{x^{LOC}_p - v(\{p\}), v(\{q\}) - x^{LOC}_q\}, \\
&= \min\{x^{LOC}_q - v(\{q\}) + x^{LOC}_p - v(\{p\}), 0\}, \\
&\leq 0. \qquad (3.19)
\end{aligned}$$

We deduce from (3.17) and (3.19) that:

$$x^*_p - v(\{p\}) \geq x^*_q - v(\{q\}). \qquad (3.20)$$

Therefore, by Lemma 3.6, (3.15) and (3.20), it holds that:

$$\begin{aligned}
\theta_1(x^*) &= \max\{x^*_k - v(\{k\}),\ k \in N\}, \\
&= \max\{x^*_p - v(\{p\}), x^*_j - v(\{j\}),\ j \in N \setminus (Q_1 \cup \{q\})\}, \\
&= \max\{x^{LOC}_p - v(\{p\}) - \frac{\triangle}{|Q_1|}, x^{LOC}_j - v(\{j\}),\ j \in N \setminus (Q_1 \cup \{q\})\}, \\
&< x^{LOC}_p - v(\{p\}), \\
&= \theta_1(x^{LOC}).
\end{aligned}$$

Hence, we conclude that $\theta(x^{LOC}) >_L \theta(x^*)$, a contradiction. □

The important second inequality in (3.14) informs us about the set of TU-games where the worth $v(N)$ can cover the sum of the individual worths, i.e. $v(N) \geq \sum_{k\in N} v(\{k\})$. We denote this set of TU-games by Γ^N_2, for which the conclusion of Proposition 3.7 still holds.

Theorem 3.8. Given a TU-game $v \in \Gamma^N_2$, the lower optimal complaint value $x^{LOC} \in I^*(v)$ is bounded below by $(v(\{i\}))_{i\in N}$.

The proof is omitted since it is similar to that of Proposition 3.7. By Theorem 3.8, the scope of the lower optimal complaint value can be restricted to the set $K' = \{x \in I^*(v) \mid \forall i \in N,\ v(\{i\}) \leq x_i\}$. To prove that the lower complaint value coincides with the CIS value, we first need the following lemma.

Lemma 3.9. Given a TU-game $v \in \Gamma_2^N$, a payoff vector $x \in K'$, and two players $l, m \in N$ such that $x_m - v(\{m\}) > x_l - v(\{l\}) \geq 0$, let $\triangle = \frac{x_m - v(\{m\}) - (x_l - v(\{l\}))}{2} > 0$ and the new payoff vector $x^* \in K'$ be defined as follows:

$$x_k^* = \begin{cases} x_k & \text{if } k \notin \{l, m\}; \\ x_l + \triangle & \text{if } k = l; \\ x_m - \triangle & \text{if } k = m. \end{cases}$$

Then, $\theta(x) >_L \theta(x^*)$.

Theorem 3.10. Given a TU-game $v \in \Gamma_2^N$ and the lower optimal complaint value $x^{LOC} \in I^*(v)$, it holds that:

(i) $x_i^{LOC} - v(\{i\})$ is a constant for all $i \in N$;

(ii) $x^{LOC} = \mathrm{CIS}(v)$.

The proofs of Lemma 3.9 and Theorem 3.10 are omitted since they are similar to those of Lemma 3.4 and Theorem 3.5, respectively. This correspondence is due in part to the duality of the EANSC and CIS values (Oishi *et al.*, 2016 [14]).

4 Axiomatic characterizations of the optimal complaint values

4.1 Characterization of the upper optimal complaint value

In this subsection we invoke two axioms. The first axiom of efficiency is classical, viz.

EFF. A value φ satisfies **efficiency** if for all $v \in \Gamma^N$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

To introduce the next axiom we require an additional notion of complaint inspired by the idea of the (pre-) kernel (Maschler and Peleg, 1966 [8], Maschler *et al.*, 1972 [9]). For a TU-game $v \in \Gamma_1^N$, the **maximal upper per capita complaint** of a player $i \in N$ against another player $j \in N$ with respect to $x \in R^n$, denoted by $h_{ij}^v(x, b^v)$, is defined as the maximal upper per capita complaint among coalitions containing i but not j, i.e.

$$h_{ij}^v(x, b^v) = \max\{\overline{e}(S, x, b^v) \mid S \subseteq N,\ i \in S \text{ and } j \notin S\}. \tag{4.1}$$

The maximal upper per capita complaint represents the maximal amount of complaint player i may express by withdrawing from x without the cooperation of player j. Thus, the maximal upper per capita complaint can be regarded as a measure of player i's negotiation power facing player j.

EUMC. A value φ satisfies the **equal upper maximal per capita complaint property** if for all $v \in \Gamma^N$ and for all $i, j \in N$, $h^v_{ij}(\varphi(v), b^v) = h^v_{ji}(\varphi(v), b^v)$.

The equal upper maximal per capita complaint property requires that any pair of players have the same maximal upper per capita complaint, i.e. their negotiation powers balance each other out.

Proposition 4.1. Given a TU-game $v \in \Gamma^N_1$, the upper optimal complaint value $x^{UOC} \in I^*(v)$ satisfies **EFF** and **EUMC**.

Proof. It is immediate that x^{UOC} satisfies **EFF**. To prove that x^{UOC} satisfies **EUMC**, note that by Theorem 3.5, the upper optimal complaint value coincides with the EANSC value. Hence:

$$\begin{aligned} h^v_{ij}(x^{UOC}, b^v) &= \max\left\{\frac{b^v(S) - x^{UOC}(S)}{|S|} \mid S \subseteq N,\, i \in S,\, j \notin S\right\}, \\ &= \max\left\{\frac{b^v(S) - \sum_{k\in S} \mathrm{EANSC}_k(v)}{|S|} \mid S \subseteq N,\, i \in S,\, j \notin S\right\}, \\ &= \frac{b^v(N) - v(N)}{n}. \end{aligned}$$

By a similar argument, we obtain $h^v_{ji}(x^{UOC}, b^v) = \frac{b^v(N)-v(N)}{n}$, which concludes the proof.□

Lemma 4.2. Given a TU-game $v \in \Gamma^N_1$, if a value φ satisfies **EUMC**, then $b^v_i - \varphi_i(v)$ is a constant for all $i \in N$.

Proof. Assume, for the sake of contradiction, that $b^v_i - \varphi_i(v)$ is not a constant for all $i \in N$, i.e. there exist two players $i, j \in N$ such that $b^v_i - \varphi_i(v) \neq b^v_j - \varphi_j(v)$. Without loss of generality, assume that $b^v_i - \varphi_i(v) > b^v_j - \varphi_j(v)$. It follows from (4.1) that:

$$h^v_{ij}(\varphi(v), b^v) = \max\left\{\frac{b^v(T) - \sum_{k\in T}\varphi_k(v) + b^v_i - \varphi_i(v)}{|T|+1} \mid T \subseteq N\backslash\{i,j\}\right\},$$

and

$$h^v_{ji}(\varphi(v), b^v) = \max\left\{\frac{b^v(T) - \sum_{k\in T}\varphi_k(v) + b^v_j - \varphi_j(v)}{|T|+1} \mid T \subseteq N\backslash\{i,j\}\right\}.$$

Since $b_i^v - \varphi_i(v) > b_j^v - \varphi_j(v)$, we obtain $h_{ij}^v(\varphi(v), b^v) > h_{ji}^v(\varphi(v), b^v)$, which is a contradiction since φ satisfies **EUMC**. $\square$

We now show that the upper optimal complaint value or, equivalently, the EANSC value, is characterized by efficiency and the equal upper maximal per capita complaint property.

Theorem 4.3. The upper optimal complaint value $x^{UOC} \in I^*(v)$ or, equivalently, the EANSC value, is the unique value satisfying **EFF** and **EUMC** on Γ_1^N.

Proof. By Proposition 4.1, it remains to show that if a value φ satisfies **EFF** and **EUMC** on Γ_1^N, then it coincides with the EANSC value. It follows from Lemma 4.2 that $b_i^v - \varphi_i(v)$ is a constant for all $i \in N$. By **EFF**, it holds that $\varphi_i(v) = b_i^v - \frac{b^v(N)-v(N)}{n}$ for all $i \in N$, which concludes the proof. $\square$

4.2 Characterization of the lower optimal complaint value

In this subsection, we invoke another axiom of maximal per capita complaint. For a TU-game $v \in \Gamma_2^N$, the **maximal lower per capita complaint** of a player $i \in N$ against another player $j \in N$ with respect to $x \in R^n$, denoted by $h_{ij}^v(x, (v(\{i\}))_{i\in N})$, is defined as the maximal lower per capita complaint among coalitions containing i but not j, i.e.

$$h_{ij}^v(x, (v(\{i\}))_{i\in N}) = \max\{\overline{e}(S, x, (v(\{i\}))_{i\in N}) \mid S \subseteq N,\, i \in S \text{ and } j \notin S\}. \tag{4.2}$$

The maximal lower per capita complaint has a similar interpretation as the maximal upper per capita complaint, viz.

ELMC. A value φ satisfies the **equal lower maximal per capita complaint property** if for all $v \in \Gamma^N$ and all $i, j \in N$, $h_{ij}^v(\varphi(v), (v(\{i\}))_{i\in N}) = h_{ji}^v(\varphi(v), (v(\{i\}))_{i\in N})$.

The equal lower maximal per capita complaint property requires that any pair of players have equal maximal lower per capita complaints.

Proposition 4.4. Given a TU-game $v \in \Gamma_2^N$, the lower optimal complaint value $x^{LOC} \in I^*(v)$ satisfies **EFF** and **ELMC**.

Proof. It is manifest that x^{LOC} satisfies **EFF**. To prove that x^{LOC} satisfies **ELMC**, note that, by Theorem 3.10, the lower optimal complaint value coincides with the CIS value. Hence:

$$\begin{aligned} h_{ij}^v(x^{LOC}, (v(\{k\}))_{k\in N}) &= \max\left\{\frac{x^{LOC}(N\backslash S) - \sum_{k\in N\backslash S} v(\{k\})}{|N\backslash S|} \mid S \subseteq N,\ i \in S,\, j \notin S\right\}, \\ &= \max\left\{\frac{\sum_{k\in N\backslash S} \mathrm{CIS}_k(v) - \sum_{k\in N\backslash S} v(\{k\})}{|N\backslash S|} \mid S \subseteq N,\ i \in S,\, j \notin S\right\}, \\ &= \frac{v(N) - \sum_{k\in N} v(\{k\})}{n}. \end{aligned}$$

By a similar argument, we obtain $h^v_{ji}(x^{LOC}, (v(\{k\}))_{k\in N}) = \frac{v(N)-\sum_{k\in N} v(\{k\})}{n}$, which concludes the proof. $\square$

Lemma 4.5. Given a TU-game $v \in \Gamma^N_2$, if a value φ satisfies **ELMC**, then $\varphi_i(v) - v(\{i\})$ is a constant for all $i \in N$.

Proof. Assume, for the sake of contradiction, that $\varphi_i(v) - v(\{i\})$ is not a constant for all $i \in N$, i.e. there exist two players $i, j \in N$ such that $\varphi_i(v) - v(\{i\}) \neq \varphi_j(v) - v(\{j\})$. Without loss of generality, assume that $\varphi_i(v) - v(\{i\}) > \varphi_j(v) - v(\{j\})$. It follows from (4.2) that:

$$h^v_{ij}(\varphi(v), (v(\{k\}))_{k\in N}) = \max\left\{\frac{\sum_{k\in T}(\varphi_k(v) - v(\{k\})) + \varphi_j(v) - v(\{j\})}{|T|+1} \mid T \subseteq N\backslash\{i, j\}\right\},$$

and

$$h_{ji}(\varphi(v), (v(\{k\}))_{k\in N}) = \max\left\{\frac{\sum_{k\in T}(\varphi_k(v) - v(\{k\})) + \varphi_i(v) - v(\{i\})}{|T|+1} \mid T \subseteq N\backslash\{i, j\}\right\}.$$

Since $\varphi_i(v)-v(\{i\}) > \varphi_j(v)-v(\{j\})$, we obtain $h_{ij}(\varphi(v), (v(\{k\}))_{k\in N}) < h_{ji}(\varphi(v), (v(\{k\}))_{k\in N})$, a contradiction since φ satisfies **ELMC**. $\square$

We now see that the lower optimal complaint value or, equivalently, the CIS value, is characterized by efficiency and the equal lower maximal per capita complaint property on Γ^N_2.

Theorem 4.6. The lower optimal complaint value $x^{LOC} \in I^*(v)$ or, equivalently, the CIS value, is the unique value satisfying **EFF** and **ELMC** on Γ^N_2.

Proof. By Proposition 4.4, it remains to show that if a value φ satisfies **EFF** and **ELMC** on Γ^N_2, then it coincides with the CIS value. It follows from Lemma 4.5 that $\varphi_i(v) - v(\{i\})$ is a constant for all $i \in N$. By **EFF**, $\varphi_i(v) = v(\{i\}) + \frac{v(N)-\sum_{k\in N} v(\{k\})}{n}$ for all $i \in N$, which concludes the proof. $\square$

5 Concluding Remarks

In this article, we have introduced two optimal complaint values. For large classes of TU-games, we have proven that the upper optimal complaint value coincides with the EANSC value, while the lower optimal complaint value coincides with the CIS value. Moreover, we have characterized these values by means of efficiency and equal maximal per capita complaint properties. The upper optimal complaint value may also coincide with the nucleolus since the coincidence between the EANSC value and the nucleolus was revealed by Driessen and Funaki (1991) [4]. However, the relationships between these equalitarian values and other solution concepts are not obvious. By lexicographically minimizing the maximal complaint vector associated with other criteria, we can obtain a variety of values for TU-games, but their links with the core and kernel remain to be determined. This work is left for future research.

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