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ADDITIVELY SEPARABLE PREFERENCES WITHOUT THE COMPLETENESS AXIOM: AN ALGEBRAIC APPROACH

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Additively separable preferences without the completeness axiom. An algebraic approach.

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Abstract

A simple mathematical result characterizing a partially ordered mean groupoid is proved and used to study the problem of additively separable preferences on preordered Cartesian product set. This means that most of the economic theory based on separable preferences - expected utility, rank-dependent expected utility, qualitative probability, discounted utility - could be generalized to the multi-utility approach.

JEL Classification: D80, D90.

Keyword: Additive utility, Separable utility, Completeness axiom, Incomplete preferences.

1 Introduction

This paper studies additively separable preferences on Cartesian products without the completeness axiom, that is, existence of additively separable utilities on preordered Cartesian products. Let, for $n \geq 2$, $X_1, \ldots, X_n$ be nonempty sets; alternatives are elements of $X = \prod_{i=1}^{n} X_i$, denoted as $x = (x_1, \ldots, x_n)$, etc. Consider a preference relation $\succsim$ which is defined as a (potentially incomplete) preorder on X. It is obvious that one cannot represent in the standard way by using a single additively separable utility function, if $\succsim$ is actually incomplete. But one may do so by means of a set of additively separable utility functions defined on X. Thus the representation notion we suggest requires one to come up with a set $\mathcal{F}$ of real functions on X such that, for all alternatives x and y,

(i) $x \succsim y \Leftrightarrow f(x) \geq f(y), \ \forall f \in \mathcal{F}$;

(ii) $f(x) = \sum_{i=1}^{n} f_i(x_i), \ \forall f \in \mathcal{F}$.

We are interested in obtaining a multi-additively separable-utility representation for preordered Cartesian products. Interest in this topic has increased during the last decade because of new developments in decision making with incomplete preferences. First, incomplete preferences leaves room for decision

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makers to remain indecisive on occasion, that is, decision makers may find themselves unable to express preferences for one alternative over another or to choose between alternatives. In the context of expected utility theory under uncertainty and risk incomplete preference relations are considered, for instance, by Bewley (2002), Galaabaatar and Karni (2013) and Dubra et al. (2004), Baucells and Shapley (2008), Galaabaatar and Karni (2012) respectively. Revealed-preference theoretic foundation for incomplete preferences are, in turn, provided by Danan (2003), Mandler (2005), Eliaz and Ok (2006). Second, additively separable preferences are frequently used in many fields of the economic theory. Applications can be found in choice under risk and uncertainty (rank-dependent models, cf., for instance, Chateauneuf and Wakker (1999), Abdellaoui (2002) and Wakker and Tversky (1993), Köbberling and Wakker (2003) respectively), case-based decision theory (cf., for instance, Gilboa et al. (2002)), inter-temporal choice (discounted utility, cf., for instance, Samuelson (1937)), and utilitarian welfare evaluations (cf., for instance, Maskin (1978)).

Additive conjoint measurement starts in the papers of Debreu (1959) for weakly ordered topological full Cartesian products, Luce and Tukey (1964)[1] for weakly ordered algebraic structures of full Cartesian products[2]. In the topological and the algebraical context, Wakker (1991, 1993) provides necessary and sufficient conditions for additive representation over weakly ordered rank-ordered subsets of full Cartesian products. The approach provided by Scott (1964) for finite subsets of Cartesian products and extended by Jaffray (1974b,a) for arbitrary countable subsets of Cartesian products applies for both weakly and preordered set. Their approach state necessary and sufficient-actually a countable set of necessary and sufficient conditions, hence hardly testable-conditions. To obtain a a multi-additively separable-utility representation, we will use the approach developed by Vind, 1991) which deals with additive representations by means of a "mean groupoid operation", i.e., an operation which assigns to each pair of points a midpoint[3]. Mean groupoids are presented in the next section.

The organization of the paper is as follows. Section 2 states representation theorem for partially ordered mean groupoids, Section 3 provides representation theorem for preordered Cartesian products, In section 4 we discuss the results and present some relevant examples. The proofs are available in the Appendix.

2 Partially ordered Commutative Mean Groupoid

A *mean groupoid* is an algebraic structure satisfying a certain kind of associative law. This structure arose from a consideration of the properties of means of pair of real numbers. Intuitively, the operation can be thought of as the arithmetic mean of or the midpoint between two elements. Aczél (1948) first introduced mean groupoids.

[1] This approach is well explained in Krantz et al. (1976).

[2] See e.g. Wakker (1988) and Luce et al. (1979) for a comparison between the topological and the algebraic framework.

[3] A comprehensive presentation can be found in Vind and Grodal (2003). We use Vind's terminology but mean groupoids are commonly called idempotent bisymmetric structure in the literature of conjoint measurement, cf., for instance Krantz et al. (1976, Chapitre 6).

Definition 1. *A commutative mean groupoid (cmg) is a set S and a function that maps each $(a,b) \in S \times S$ into an element $x \circ y$ in S such that for every $a,b,c,d \in S$,*

MG.1. Idempotent, $a \circ a = a$,

MG.2. Commutative, $a \circ b = b \circ a$,

MG.3. Bisymmetric, $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$.

A *partially ordered commutative mean groupoid* (pocmg) $(S, \circ, \succeq)$ is a cmg $(S, \circ)$ and a partial order (reflexive, antisymmetric and transitive) $\succeq$ on S such that, for all $a,b,c \in S$, $a \succeq b \Leftrightarrow a \circ c \succsim b \circ c$. A pocmg is called bounded if there exists $u, v \in S$, such that for all $a \in S$, $v \succeq a \succeq u$. In the sequel we always suppose that $v \succ u$, otherwise $\succeq$ would be trivial. A pocmg is called Archimedean iff for all $a,b,c \in S$, $(\forall n \geq 1, b \succeq na \circ c) \Rightarrow b \succeq a$, where $na \circ c = a \circ (a \circ \ldots \circ (a \circ c))$.

We are now able to state and prove the representation theorem for partially ordered mean groupoids. In the sequel this theorem will be applied to preordered Cartesian products set $(X, \succsim)$ provided it has enough extra properties to define a mean groupoid operation on $(X/\sim, \succsim)$.

Theorem 1. *Let $(S, \circ, \succeq)$ be an Archimedean, bounded, partially ordered commutative mean groupoid. Then there exists a nonempty set of functions $\mathcal{F} \subseteq [0,1]^S$ such that for all $a, b \in S$,*

(i) $a \succeq b \Leftrightarrow f(a) \geq f(b), \ \ \forall f \in \mathcal{F}$;

(ii) $f(a \circ b) = \frac{f(a)+f(b)}{2}, \ \ \forall f \in \mathcal{F}$;

(iii) $f(v) = 1$ and $f(u) = 0$, $\forall f \in \mathcal{F}$.

In other words, $(S, \circ, \succeq)$ is isomorphic to a sub-mean groupoid of $[0,1]^{\mathcal{F}}$ (with the pointwise ordering), for some nonempty set $\mathcal{F}$.

We now turn to generalize the uniqueness part of the classical representation theorem of mean groupoid in the present context. Let $(S, \circ, \succeq)$ be an Archimedean, bounded, pocmg and let $\mathcal{F} \subseteq [0,1]^S$ be a nonempty set of functions which represents $\succeq$ in the sense of (i). We view $\mathcal{F}$ as a subset of the real vector space $\mathbb{R}^S$ of all real-valued functions on S. In addition, we equip $\mathbb{R}^S$ with the product topology, so that it becomes a linear topological space and more specifically a locally convex Hausdorff space. We define $cl(Conv(\mathcal{F}))$ as the closure, with respect to the topology on $\mathbb{R}^S$, of the convex hull generated by $\mathcal{F}$. We have the following uniqueness result.

Proposition 1. *Let $(S, \circ, \succeq)$ be an Archimedean, bounded, partially ordered commutative mean groupoid. $\mathcal{F}, \mathcal{G} \subseteq [0,1]^S$ represent $\succeq$ in the sense of (i), if and only if $cl(Conv(\mathcal{F})) = cl(Conv(\mathcal{G}))$.*

3 Additively separable preferences

3.1 Intuitive conditions for additive conjoint measurement

In this subsection we present the most important characterizing conditions, coordinate independence, coordinate independence consistency and the Thomsen condition. Let, for $n \geq 2$, $X_1, \dots, X_n$ be nonempty sets; alternatives are elements of $X = \prod_{i=1}^n X_i$, denoted as $x = (x_1, \dots, x_n)$, etc. We write $x_{-i}v_i$ for x with x_i replaced by v_i, and, for $i \neq j$, $x_{-i,j}v_iw_j$ for x with x_i replaced by v_i, x_j replaced by w_j. We call $\succsim$ a preorder on X if it is both reflexive and transitive. As usual, $\succ$ denotes the asymmetric part, $\sim$ the symmetric part.

Definition 2. *We say that $\succsim$ satisfies coordinate independence if for all $i = 1, \dots, n$, for all x, y and for all v_i, w_i,*

$$x_{-i}v_i \succsim y_{-i}v_i \Leftrightarrow x_{-i}w_i \succsim y_{-i}w_i.$$

So, if two alternatives have a common coordinate, then by coordinate independence the preference is unaffected when that coordinate is changed into any other common coordinate. Perhaps the most important feature of a coordinate independent relation is that it induces a natural ordering on $\prod_I X_i$ for every subset I of $\{1, \dots, n\}$.

Definition 3. *We say that $\succsim$ satisfies coordinate independence consistency if for all $i = 1, \dots, n$, for all x, y and for all v_i, w_i such that $x_{-i}v_i \succsim y_{-i}w_i$,*

$$y_{-i}v_i \succsim x_{-i}w_i \Rightarrow y_{-i}w_i \succsim x_{-i}w_i.$$

Coordinate independence consistency is implied by coordinate independence under completeness. It is, in turn, a necessary condition for a multi-additively separable-utility representation. This condition states that there is no "gap" in a coordinate independent preordered Cartesian product.

Definition 4. *Let $n = 2$. We say that $\succsim$ satisfies the Thomsen condition if for all $x_1, y_1, v_1 \in X_1$ and $x_2, y_2, v_2 \in X_2$,*

$$x_1v_2 \sim v_1y_2 \text{ and } v_1x_2 \sim y_1v_2 \Rightarrow x_1x_2 \sim y_1y_2.$$

3.2 Technical conditions of additive conjoint measurement

In this subsection we give technical definitions,

Definition 5. *We say $\succsim$ is essentially bounded if there exist $\bar{x}, \underline{x}$ such that for all x, $\bar{x} \succsim x \succsim \underline{x}$ and for all $i, j = 1, \dots, n$, $\underline{x}_{-i}\bar{x}_i \sim \underline{x}_{-j}\bar{x}_j$.*

Definition 6. *We say $\succsim$ is bisectable relatively to the bounds (bisectable r.b.) if $\succsim$ is essentially bounded and, for all x, for all $i, j = 1, \dots, n$, $i \neq j$, there exists v_i, w_j such that $x_{-i,j}v_iw_j \sim x$ and $\underline{x}_{-i}v_i \sim \underline{x}_{-j}w_j$.*

Definition 7. *We say $\succsim$ satisfies restricted solvability if, for all $x_{-i}u_i \succsim y \succsim x_{-i}w_i$, there exists a v_i such that $x_{-i}v_i \sim y$.*

Aside from restricted solvability which is standard in additive conjoint measurement, the other two definitions require clarification. Under completeness and restricted solvability, provided that all factors are essential, there exists at least (actually, there exist many) a substructure in X where we can find a representation. This representation is then extended to the whole of X. For example in the two factors case, under completeness, if there exist $x_1, y_1 \in X_1$, and $x_2, y_2 \in X_2$ such that $x_1y_2 \succ y_1y_2$ and $y_1x_2 \succ y_1y_2$, then restricted solvability yields to an essentially bounded structure whenever $x_1y_2 \succsim y_1x_2$ or $y_1x_2 \succsim x_1y_2$. When $\succsim$ is no longer complete, such substructures do not necessarily exist. Thereby, $\succsim$ is essentially bounded states that X has the profile of this substructure. The bisectable condition is a richness condition, it states that there are enough elements to define a midpoint operation in the whole structure. This assumption implies, in the presence of the other axiom, that the factor X_i are infinite and are dense in the way the real or the dyadic rational number are. Next we give the Archimedean axiom for a multi-additively separable-utility representability.

Definition 8. *We say $\succsim$ is Archimedean if, for all $i, j = 1, \ldots, n$, $i \neq j$, for all x, y, for all $(v_i^k)_{k\geq 1}$, $(v_j^k)_{k\geq 1}$, $(w_i^k)_{k\geq 1}$, $(w_j^k)_{k\geq 1}$ such that for all $k \geq 1$, $x \sim x_{-i,j}v_i^k v_j^k$ and $x_{-i,j}v_i^k w_j^k \sim x_{-i,j}w_i^k v_j^k \sim x_{-i,j}w_i^{k+1}w_j^{k+1}$;*

$$\forall k \geq 1,\ y \succsim x_{-i,j}w_i^{k+1}w_j^{k+1} \Rightarrow y \succsim x.$$

This is obviously necessary for a multi-additively separable-utility representability, since the preferences involved imply that $x_{-i,j}w_i^{k+1}w_j^{k+1}$ converges to x.

3.3 The Main Results

The first theorem of this subsection deals with the case of two factors X_1 and X_2.

Theorem 2. *Suppose X_1 and X_2 are nonempty sets. Let $\succsim$ be an essentially bounded binary relation on $X_1 \times X_2$ that satisfies restricted solvability and the bisectable condition. The binary relation $\succsim$ is a preorder satisfying the Archimedean axiom, coordinate independence, coordinate independence consistency and the Thomsen condition if and only if there exists a nonempty set of functions $\mathcal{F}$ such that, for all alternatives x and y,*

(i) $x \succsim y \Leftrightarrow f(x) \geq f(y),\ \ \forall f \in \mathcal{F}$;

(ii) $f(x) = f(x_1, \underline{x}_2) + f(\underline{x}_1, x_2),\ \ \forall f \in \mathcal{F}$;

(iii) $f(\underline{x}) = 0$ *and* $f(\bar{x}) = 1$, $\forall f \in \mathcal{F}$.

Further, $\mathcal{F}, \mathcal{G}$ represent $\succeq$ in the sense of (i), if and only if $cl(Conv(\mathcal{F})) = cl(Conv(\mathcal{G}))$.

Next when there are three or more factors, as in the case where $\succsim$ is complete, the Thomsen condition can be removed. We have the following result.

Theorem 3. *Suppose $X_1, \ldots, X_n$ are nonempty sets. Let $\succsim$ be an essentially bounded binary relation on $\prod_{i=1}^n X_i$ with $n \geq 3$ that satisfies restricted solvability and the bisectable condition. The binary relation $\succsim$ is a preorder satisfying the Archimedean axiom, coordinate independence and coordinate independence consistency if and only if there exists a nonempty set of functions $\mathcal{F}$ such that, for all alternatives x and y,*

(i) $x \succsim y \Leftrightarrow f(x) \geq f(y), \ \ \forall f \in \mathcal{F}$;

(ii) $f(x) = \sum_{i=1}^n f(\underline{x}_1, \ldots, \underline{x}_{i-1}, x_i, \underline{x}_{i+1}, \ldots, \underline{x}_n), \ \ \forall f \in \mathcal{F}$;

(iii) $f(\underline{x}) = 0$ *and* $f(\bar{x}) = 1$, $\forall f \in \mathcal{F}$.

Further, $\mathcal{F}, \mathcal{G}$ represent $\succeq$ in the sense of (i), if and only if $cl(Conv(\mathcal{F})) = cl(Conv(\mathcal{G}))$.

4 Discussion

A first contribution of Theorems 2 and 3 is one of a mathematical nature. It allows to give conditions for the existence of a multi-additively separable-utility representation for preordered full Cartesian products. In a more general setting, if a preordered full Cartesian product is covered by a system of equivalent essentially bounded subsets which overlap on bounds, each of which satisfies the required axioms. If a multi-additively separable-utility representation holds over each of the subsets, then by the uniqueness of the representation we know that the representations over two subset can be chosen so that they coincide over the common bound. In this way, the representation can be extended throughout the preordered full Cartesian product. In this sense, our results extend the actual results of additive conjoint measurement on full Cartesian product with a "tight" weak order structure. To view it, note that our axioms are fulfilled in the topological approach of Debreu (1959), Wakker (1988) and in some infinite structures of the algebraic approach of Luce and Tukey (1964), Krantz et al. (1976), provided that the order is bounded. Nevertheless, keeping the "tight" structural conditions, our results are restrictive as illustrated in the following example.

Example 1. *Let $X_1 = X_2 = [0, 1]$. Let a preorder $\succsim$ on $X_1 \times X_2$ be defined by $x \succsim y$ if and only if $x_1 + x_2 \geq y_1 + y_2$ and $x_1^2 + x_2^2 \geq y_1^2 + y_2^2$. It is easy to verify that $\succsim$ is an essentially bounded preorder satisfying the Archimedean axiom, coordinate independence, coordinate independence consistency and the Thomsen condition. However, $\succsim$ does not satisfy restricted solvability and the bisectable condition.*

Our approach is partially satisfactory because, we require that $\succsim$ leads to the same mean groupoid operation on $(X/\sim, \succsim)$. Note that, in the previous example, each representation defines a distinct mean groupoid operation. Hence a satisfactory approach requires to not only extend an arbitrary preorder but to extend an arbitrary preorder together with the underlying structure.

To conclude, we give a slight application of our result. Following Vind and Grodal (2003) or Krantz et al. (1976), Theorem 1 can be applied directly to qualitative probability and discounted utility as illustrated in the following example.

Example 2. *Let X be a nonempty set, $\mathcal{E}$ be an algebra of sets on X and $\succsim$ be a preorder on $\mathcal{E}$ which satisfies $X \succ \emptyset$, $X \succsim A \succsim \emptyset$ for all $A \in \mathcal{E}$ and $A \cap C = B \cap C = \emptyset$ implies $A \succsim B \Leftrightarrow A \cup C \succsim B \cup C$. If $\succsim$ is polarizable, that is, for all $A \in \mathcal{E}$, there exist A', A'' such that $A' \sim A''$, $A' \cup A'' = A$ and $A' \cap A'' = \emptyset$ together with an Archimedean axiom, then there exists a nonempty set of probability functions $\mathcal{P}$ on $\mathcal{E}$ such that, for all A and B,*

$$A \succsim B \Leftrightarrow P(A) \geq P(B), \ \ \forall P \in \mathcal{P}$$

Appendix

Proofs from section 2

We begin with collecting a group of simple results concerning pocmg that will prove useful in the following.

Definition 9. *A symmetric commutative mean groupoid (scmg) $(S, \circ)$ is a cmg $(S, \circ)$ which satisfies*

MG.4. $\exists e \in S, \ \forall a \in S, \ \exists -a, \ a \circ -a = e.$

Lemma 1. *A pocmg $(S, \circ, \succeq)$ is isomorphic to a subset of a partially ordered symmetric commutative mean groupoid $(S', *, \succeq')$. Moreover, if $(S, \circ, \succeq)$ is Archimedean and bounded then $(S', *, \succeq')$ so is.*

Proof. Let $(S, \circ, \succeq)$ be a pocmg. Define $\succeq'$ on S^2 as follows: for all $a, b, c, d \in S$,

$$(a, b) \succeq' (c, d) \ \ iff \ \ a \circ d \succeq c \circ b.$$

It is straightforward to prove that $\succeq'$ is a preorder. Define:

$[a, b] = \{(a', b') | a', b' \in S, \ \ (a', b') =' (a, b)\}$,

$S' = S^2 / =' ,$

$* \ \ : [a, b] * [c, d] = [a \circ c, b \circ d]$,

$[0] = [a, a]$, $a \in S$,

$-[a, b] = [b, a]$.

There is no difficulty to prove that $*$, $[0]$ and $-[a, b]$ are well defined, that $(S', *, \succeq')$ is a partially ordered symmetric commutative mean groupoid, and that $(S, \circ, \succeq)$ is isomorphic to a subset of $(S', *, \succeq')$. To complete the proof we need only show that if $(S, \circ, \succeq)$ is Archimedean and bounded then $(S', *, \succeq')$ so is.

First, suppose that $(S, \circ, \succeq)$ is Archimedean and that $\forall n \geq 1, [a,b] \succeq' n[c,d] * [e,f]$. We show that $[a,b] \succeq' [c,d]$.

$$\begin{aligned}
&\forall n \geq 1,\ [a,b] \succeq' n[c,d] * [e,f] \\
\Rightarrow\ &\forall n \geq 1,\ n[d,c] * [f,e] \succeq' [b,a] \\
\Rightarrow\ &\forall n \geq 1,\ [e,f] * (n[d,c] * [f,e]) \succeq' [e,f] * [b,a].
\end{aligned}$$

By induction, it is easy to prove that $[e,f] * (n[d,c] * [f,e]) = n([e,f] * [d,c]) * [0]$, therefore

$$\begin{aligned}
&\forall n \geq 1,\ n([e,f] * [d,c]) * [0] \succeq' [e,f] * [b,a] \\
\Rightarrow\ &\forall n \geq 1,\ [f,e] * [a,b] \succeq' n([f,e] * [c,d]) * [0] \\
\Rightarrow\ &\forall n \geq 1,\ [f \circ a, e \circ b] \succeq' n([f \circ c, e \circ d] * [e \circ d, e \circ d] \\
\Rightarrow\ &\forall n \geq 1,\ [f \circ a, e \circ b] \succeq' [n(f \circ c) \circ (e \circ d), e \circ d] \\
\Rightarrow\ &\forall n \geq 1,\ (f \circ a) \circ (e \circ d) \succeq (n(f \circ c) \circ (e \circ d)) \circ (e \circ b) \\
\Rightarrow\ &\forall n \geq 1,\ (f \circ a) \circ (e \circ d) \succeq (n[(f \circ c) \circ (e \circ b)] \circ [(e \circ d) \circ (e \circ b)]
\end{aligned}$$

It follows by the Archimedean hypothesis,

$$\begin{aligned}
&(f \circ a) \circ (e \circ d) \succeq (f \circ c) \circ (e \circ b) \\
\Rightarrow\ &(f \circ e) \circ (a \circ d) \succeq (f \circ e) \circ (c \circ b) \\
\Rightarrow\ &(a \circ d) \succeq (c \circ b) \\
\Rightarrow\ &[a,b] \succeq' [c,d].
\end{aligned}$$

Hence, $(S', *, \succeq')$ is Archimedean.

Second, suppose that $(S, \circ, \succeq)$ is bounded, then by definition for all $a, b \in S$, $v \succeq a \succeq u$ and $v \succeq b \succeq u$. Therefore, $v \circ b \succeq a \circ b \succeq u \circ b$ and $v \circ a \succeq a \circ b \succeq u \circ a$. It follows from transitivity that $v \circ b \succeq u \circ a$ and that $v \circ a \succeq u \circ b$ and by definition that $[v,u] \succeq' [a,b] \succeq' -[v,u]$. Moreover $v \succ u$ implies $[v,u] \succ' -[v,u]$. That is $(S', *, \succeq')$ is bounded, as required. □

In a partially ordered scmg, it is clear that $a \succeq b$ is equivalent to $a \circ (-b) \succeq e$, that e is unique and equals to its inverse. An element a of S is called positive element if $a \succeq e$. The set of positive element is denoted by S^+. Note that the boundedness property becomes: $\exists v \in S,\ v \succ e,\ \forall a \in S,\ v \succeq a \succeq -v$. With a slight abuse of notation, we denote by (S, v) a bounded partially ordered symmetric commutative mean groupoid.

In the sequel, instead of the notation $a \circ b$, we will use the arithmetic mean notation for a symmetric commutative mean groupoid. That is, $\frac{a+b}{2}$ is used to indicate the result of the operation $a \circ b$, 0 is used for the identity element, $-a$ is used for the inverse element, and $\frac{k}{2^n} a$ is used to indicate the arbitrary product of n terms of a and 0 with $k = 0, \ldots, 2^n$ the effective weight of a in the product.

Definition 10. *Let (S, v) be a bounded partially ordered symmetric commutative mean groupoid. A normalized numerical representation f on (S, v) is any map $f : S \to \mathbb{R}$ such that $f(\frac{a+b}{2}) = \frac{f(a)+f(b)}{2}$, $f(S^+) \subseteq \mathbb{R}^+$ and $f(v) = 1$.*

Lemma 2. *Let (S, v) be a bounded partially ordered symmetric commutative mean groupoid, let T be a symmetric sub-mean groupoid of S, and let $x \in S$. Let f a normalized numerical representation on (T, v), and set*

$$\begin{aligned} p =& sup\left\{ \frac{2^n}{k} f(y) \;\middle|\; y \in T;\ n \in \mathbb{N};\ k = 0, \dots, 2^n;\ \frac{k}{2^n} x \succeq y \right\}, \\ r =& inf\left\{ \frac{2^m}{j} f(z) \;\middle|\; z \in T;\ m \in \mathbb{N};\ j = 0, \dots, 2^m;\ z \succeq \frac{j}{2^m} x \right\}. \end{aligned}$$

Then,

(a) $1 \geq r \geq p \geq -1$.

(b) If there exists a normalized numerical representation g on the symmetric mean groupoid generated by $\{T, x\}$ extending f, then $r \geq g(x) \geq p$.

(c) If $r \geq q \geq p$, then f extends to a normalized numerical representation g on the symmetric mean groupoid generated by $\{T, x\}$ such that $g(x) = q$.

Proof. (a) If there exist $y, z \in T$, $n, m \in \mathbb{N}$, $k \in \{0, \dots, 2^n\}$ and $j \in \{0, \dots, 2^m\}$ such that $z \succeq \frac{j}{2^m} x$ and $\frac{k}{2^n} x \succeq y$, then $\frac{k}{2^n} z \succeq \frac{kj}{2^{m+n}} x \succeq \frac{j}{2^m} y$ and so $\frac{2^m}{j} f(z) \geq \frac{2^n}{k} f(y)$, whence $r \geq p$. By the boundedness hypothesis, note that $1 \geq r$ and $p \geq -1$. Thus $1 \geq r \geq p \geq -1$.

(b) If $y, z \in T$, $n, m \in \mathbb{N}$, $k \in \{0, \dots, 2^n\}$ and $j \in \{0, \dots, 2^m\}$ such that $z \succeq \frac{j}{2^m} x$ and $\frac{k}{2^n} x \succeq y$, then $f(z) = g(z) \geq \frac{j}{2^m} g(x)$ and $\frac{k}{2^n} g(x) \geq g(y) = f(y)$, whence $\frac{2^m}{j} f(z) \geq g(x) \geq \frac{2^n}{k} f(y)$. Thus $r \geq g(x) \geq p$.

(c) Let U be the symmetric sub-mean groupoid generated by $\{T, x\}$, it is not hard to see that

$$U = \left\{ \frac{\sum_{i=1}^{p} \alpha_i a_i + (\beta_+ - \beta_-) x}{2^n} \;\middle|\; a_i \in T,\ n, \alpha_i, \beta_+, \beta_- \geq 0,\ \sum_{i=1}^{p} \alpha_i + \beta_+ + \beta_- = 2^n \right\}.$$

U is a mean groupoid, it includes T ($\beta_+ = \beta_-$), it includes the symmetric sub-mean groupoid generated by $\{-x, x\}$ ($a_i = 0$, for all i), and each element has an inverse (replace a_i by $-a_i$ and exchange β_+, β_-). We show that

$$\frac{\sum_{i=1}^{p} \alpha_i a_i + (\beta_+ - \beta_-) x}{2^n} \succeq 0 \Rightarrow \sum_{i=1}^{p} \alpha_i f(a_i) + (\beta_+ - \beta_-) q \geq 0.$$

If $\beta_+ - \beta_- = 0$, then it is true as f is a normalized numerical representation on

T. Suppose then, that $\beta_+ - \beta_- \neq 0$.

$$\begin{aligned}
&\frac{\sum_{i=1}^{p} \alpha_i a_i + (\beta_+ - \beta_-)x}{2^n} \succeq 0 \\
\Rightarrow &\frac{1}{2}\left(\frac{\sum_{i=1}^{p} \alpha_i a_i + (\beta_+ - \beta_-)x}{2^n} + \frac{(\beta_- - \beta_+)x}{2^n}\right) \succeq \frac{1}{2}\left(0 + \frac{(\beta_- - \beta_+)x}{2^n}\right) \\
\Rightarrow &\frac{\sum_{i=1}^{p} \alpha_i a_i}{2^{n+1}} \succeq \frac{(\beta_- - \beta_+)x}{2^{n+1}} \\
\Rightarrow &\frac{\sum_{i=1}^{p} \alpha_i a_i}{2^n} \succeq \frac{(\beta_- - \beta_+)x}{2^n}
\end{aligned}$$

If $\beta_- > \beta_+$, then $\frac{2^n}{\beta_- - \beta_+} f\left(\frac{\sum_{i=1}^{p} \alpha_i a_i}{2^n}\right) \geq r \geq q$, rearranging

$$\sum_{i=1}^{p} \alpha_i f(a_i) + (\beta_+ - \beta_-)q \geq 0$$

The proof for $\beta_+ > \beta_-$ is similar. Therefore f extends to a well-defined normalized numerical representation g on the symmetric mean groupoid generated by $\{T, x\}$ such that $g(x) = q$. □

Lemma 3. *Let (S, v) be a bounded partially ordered symmetric commutative mean groupoid. Then there exists a normalized numerical representation f on (S, v).*

Proof. Let (S, v) be a bounded partially ordered commutative mean groupoid. Since S is bounded, $v \succ 0$, and then the symmetric sub-mean groupoid $S_{-v,v}$ generated by $\{-v, v\}$ is simply ordered. Hence, $S_{-v,v} \cap S^+ = S^+_{-v,v}$. Thus there exists a normalized numerical representation f_0 on $(S_{-v,v}, v)$ with values on the dyadic rationals in the interval from -1 to 1.

Let $\mathcal{D}(g)$ denote the domain of a function g. Let $\mathcal{J}$ denote the collection of all pairs $(\mathcal{D}(g), g)$ for which the following is true: $\mathcal{D}(g)$ is a symmetric sub-mean groupoid of S containing $S_{-v,v}$ and g is a normalized numerical representation on $(\mathcal{D}(g), v)$ extending f_0. Clearly $\mathcal{J}$ is nonempty since it includes f_0. Define $\succsim$ on $\mathcal{J}$ as follows: if $(\mathcal{D}(g), g), (\mathcal{D}(h), h) \in \mathcal{J}$, then $(\mathcal{D}(h), h) \succsim (\mathcal{D}(g), g)$ if and only if $\mathcal{D}(g) \subseteq \mathcal{D}(h)$ and h extends g. It is trivial to see that $\succsim$ is a partial order for which any simply ordered nonempty subset of $\mathcal{J}$ has an upper bound in $\mathcal{J}$. Therefore, by Zorn's lemma, there exists a maximal element $(\mathcal{D}(f), f)$, and we need only show that $\mathcal{D}(f) = S$.

Suppose that, on the contrary, there exists an element $x \in S \setminus \mathcal{D}(f)$. Lemma 2 shows that f extends to a normalized numerical representation g on the symmetric mean groupoid generated by $\{\mathcal{D}(f), x\}$. So $(\mathcal{D}(f), f)$ is not maximal in $\mathcal{J}$, contrary to choice. Thus $\mathcal{D}(f) = S$. $\square$

Lemma 4. *Let (S, v) be an Archimedean, bounded, partially ordered symmetric commutative mean groupoid. Suppose that $n \geq 0$ and $k \in \{1, \ldots, 2^n\}$, then*

$$\frac{k}{2^n} x \succeq 0 \Rightarrow x \succeq 0.$$

Proof. Let (S, v) be an Archimedean, bounded, partially ordered symmetric commutative mean groupoid. Suppose that $n \geq 0$, $k \in \{1, \ldots, 2^n\}$, and that $\frac{k}{2^n} x \succeq 0$. For $k = 1$, we have $\frac{k}{2^n} x = n0 \circ x$, by MG.1 and monotonicity, it is true that $x \succeq 0$. Next, suppose that $k \geq 2$. By monotonicity and the boundedness assumption, it is true that

$$\forall p, m \geq 0, \ \frac{k}{2^{n+p}} x \succeq -\frac{1}{2^m} v.$$

If $k \geq 2$ and $p \geq 0$, then $\frac{2^{p+1}-k}{2^{n+p}} x$ lies in the sub-mean groupoid generated by $\{0, \frac{k}{2^n} x\}$ which implies that $\frac{2^{p+1}-k}{2^{n+p}} x \succeq 0$. Whence, for all $m \geq 1$

$$\frac{1}{2^n} x = \frac{1}{2} \left(\frac{k}{2^{n+p}} x + \frac{2^{p+1} - k}{2^{n+p}} x \right) \succeq -\frac{1}{2^m} v,$$

which, in turn, is equivalent to

$$\forall m \geq 1, \ n0 \circ x \succeq m0 \circ (-v).$$

Then $n0 \circ x \succeq 0$ because (S, v) is Archimedean. Therefore $x \succeq 0$. $\square$

Lemma 5. *Let (S, v) be an Archimedean, bounded, partially ordered symmetric commutative mean groupoid, and let $x \in S$. Then $f(x) \geq 0$ for all normalized numerical representations f on (S, v) if and only if $nx \circ v = \frac{(2^n-1)x+v}{2^n} \succeq 0$ for all $n \geq 0$.*

Proof. Note that the set of all normalized numerical representations on (S, v) is nonempty by Lemma 3.

First assume that $nx \circ u = \frac{(2^n-1)x+v}{2^n} \succeq 0$ for all $n \geq 0$. Given any normalized numerical representation f on (S, v), we then have

$$(2^n - 1) f(x) + 1 \geq 0$$

for all n, whence $f(x) \geq -\frac{1}{2^n-1}$ for all n, and thus $f(x) \geq 0$.

Conversely, assume that $f(x) \geq 0$ for all normalized numerical representations f on (S, v). Set

$$p = sup \left\{ \frac{i}{k} 2^{n-m} \;\middle|\; n, m \in \mathbb{N}; \ k = 0, \ldots, 2^n; \ i = -2^m, \ldots, 2^m; \ \frac{k}{2^n} x \succeq \frac{i}{2^m} v \right\}.$$

Since S is bounded, recall that there exists a normalized numerical representation f_0 on $(S_{-v,v}, v)$ with values on the dyadic rationals in the interval from -1 to 1, and we observe that

$$p = sup\left\{\frac{2^n}{k} f_0(y) \;\middle|\; y \in S_{-v,v};\; n \in \mathbb{N};\; k = 0, \dots, 2^n;\; \frac{k}{2^n} x \succeq y\right\}.$$

We can deduce, by the same reasoning as in Lemma 3, that there exists a normalized numerical representation f on (S, v) such that $f(x) = p$, and hence $p \geq 0$. Now given any $j \in \mathbb{N}$, there must exist $n, m \in \mathbb{N}$, $k \in \{0, \dots, 2^n\}$ and $i \in \{-2^m, \dots, 2^m\}$ such that $\frac{i}{k} 2^{n-m} \geq -\frac{1}{2^j-1}$ and $\frac{k}{2^n} x \succeq \frac{i}{2^m} v$. Then

$$\frac{i}{2^m} \frac{2^j - 1}{2^j} \geq -\frac{k}{2^n} \frac{1}{2^j},$$

whence

$$\frac{k}{2^n} \frac{2^j - 1}{2^j} x \succeq \frac{i}{2^m} \frac{2^j - 1}{2^j} v \succeq -\frac{k}{2^n} \frac{1}{2^j} v,$$

and therefore

$$\frac{k}{2^n} \left(\frac{(2^j - 1)x + v}{2^j}\right) \succeq 0.$$

Thus, by Lemma 4, we can conclude that $jx \circ v \succeq 0$ for all $j \geq 0$. □

We are now prepared to prove Theorem 1.

Proof of Theorem 1. Let $(S, \circ, \succeq)$ be an Archimedean, bounded, partially ordered commutative mean groupoid. By Lemma 1, $(S, \circ, \succeq)$ is isomorphic to a subset of an Archimedean, bounded, partially ordered symmetric commutative mean groupoid $(S', *, \succeq')$ with bound $[v, u]$. Let $\mathcal{F}'$ be the set of normalized numerical representations on $(S', [v, u])$ which is nonempty by Lemma 3. Lemma 5 prove that

$$f'([a, b]) \geq 0,\ \forall f' \in \mathcal{F}' \Leftrightarrow n[a, b] * [v, u] \succeq' [0],\ \forall n \geq 0.$$

It is straightforward to show that the Archimedean assumption implies the following property

$$\forall n \geq 0,\ n[a, b] * [v, u] \succeq' [0] \Rightarrow [a, b] \succeq' [0].$$

And therefore

$$f'([a, b]) \geq 0,\ \forall f' \in \mathcal{F}' \Leftrightarrow [a, b] \succeq' [0].$$

For all $f' \in \mathcal{F}'$ define f on S as follows: for all $a \in S$, $f(a) = f'([a, u])$; and define $\mathcal{F}$ be the collection of all the function f defined in this way. As $\mathcal{F}'$ is the set of normalized numerical representations on $(S', [v, u])$, then for all $f \in \mathcal{F}$, $f(u) = 0$ and $f(v) = 1$. Moreover, it is straightforward to check that for all $a, b \in S$,

(i) $a \succeq b \Leftrightarrow f(a) \geq f(b),\ \ \forall f \in \mathcal{F}$;

(ii) $f(a \circ b) = \frac{f(a)+f(b)}{2},\ \ \forall f \in \mathcal{F}$.

□

Proof of Proposition 1. Since necessity is obvious, we shall prove here only the sufficiency. Let $(S, \circ, \succeq)$ be an Archimedean, bounded, partially ordered commutative mean groupoid. By Lemma 1, $(S, \circ, \succeq)$ is isomorphic to a subset of an Archimedean, bounded, partially ordered symmetric commutative mean groupoid $(S', *, \succeq')$. Let $\mathbb{R}^{S'}$ be the set of all real-valued functions on S' equipped with the product topology, so that it becomes a locally convex Hausdorff space. Let $\mathcal{F}' \subseteq [-1,1]^S$ (respectively $\mathcal{G}'$) be the set of normalized numerical representations on S' defined by $\mathcal{F}$ (respectively $\mathcal{G}$). Suppose that there exists a normalized numerical representations g' such that $g' \in cl(Conv(\mathcal{G}')) \setminus cl(Conv(\mathcal{F}'))$. $\{g'\}$ and $cl(Conv(\mathcal{F}'))$ are disjoint nonempty compact convex subsets of $\mathbb{R}^{S'}$. Then, by the Hahn-Banach Theorem , there exists $[a,b] \in S'$ such that

$$sup\,\{h([a,b])|h = g'\} < 0 \leq inf\,\{h([a,b])|h \in \mathrm{cl}(\mathrm{Conv}(\mathcal{F}'))\}\,.$$

Whence, we get $g'([a,b]) < 0$ and $f'([a,b]) \geq 0$ for all $f' \in \mathcal{F}'$, which is a contradiction. Therefore, if $\mathcal{F}, \mathcal{G} \subseteq [0,1]^S$ represent $\succeq$ in the sense of (i), then $cl(Conv(\mathcal{F})) = cl(Conv(\mathcal{G}))$. $\square$

Proofs from section 3

Proof of Theorem 2. $\succsim$ is a preorder so that $\sim$ is an equivalence. Let $S = X_1 \times X_2/\sim$ be the set of equivalence classes of $X_1 \times X_2$ under $\sim$. By a slight abuse of notation, we let $\succsim$ be the induced partial order on S. We define $\circ$ on S by letting $[x_1x_2] \circ [y_1y_2] = [z_1z_2]$ if there exist $z_1, t_1 \in X_1$, $z_2, t_2 \in X_2$ such that $z_1\underline{x}_2 \sim \underline{x}_1t_2$, $t_1\underline{x}_2 \sim \underline{x}_1z_2$, $z_1t_2 \sim x_1x_2$ and $t_1z_2 \sim y_1y_2$. We shall now prove that $(S, \circ, \succsim)$ is an Archimedean, bounded, partially ordered commutative mean groupoid (proven below in steps labelled from 1 to 7).

1. $\circ$ is well defined By the bisectable condition, $\circ$ is defined on $S \times S$. If there exist $z_1, t_1, z_1', t_1' \in X_1$, $z_2, t_2, z_2', t_2' \in X_2$ such that $z_1\underline{x}_2 \sim \underline{x}_1t_2$, $t_1\underline{x}_2 \sim \underline{x}_1z_2$, $z_1'\underline{x}_2 \sim \underline{x}_1t_2'$, $t_1'\underline{x}_2 \sim \underline{x}_1z_2'$, $z_1t_2 \sim z_1't_2' \sim x_1x_2$ and $t_1z_2 \sim t_1'z_2' \sim y_1y_2$. By the Thomsen condition, $z_1'\underline{x}_2 \sim \underline{x}_1t_2'$ and $z_1\underline{x}_2 \sim \underline{x}_1t_2$ imply $z_1't_2 \sim z_1t_2'$, as $z_1t_2 \sim z_1't_2'$, coordinate independence consistency implies that $z_1t_2 \sim z_1't_2$. Identically, $t_1z_2 \sim t_1z_2'$. By coordinate independence, $z_1z_2 \sim z_1'z_2'$, so $\circ$ is well defined.

2. $\circ$ is idempotent By definition, $[x_1x_2] \circ [x_1x_2] = [x_1x_2]$.

3. $\circ$ is commutative By definition, $[x_1x_2] \circ [y_1y_2] = [z_1z_2]$ if there exist $z_1, t_1 \in X_1$, $z_2, t_2 \in X_2$ such that $z_1\underline{x}_2 \sim \underline{x}_1t_2$, $t_1\underline{x}_2 \sim \underline{x}_1z_2$, $z_1t_2 \sim x_1x_2$ and $t_1z_2 \sim y_1y_2$. That is, $[y_1y_2] \circ [x_1x_2] = [t_1t_2]$. By the Thomsen condition, $z_1\underline{x}_2 \sim \underline{x}_1t_2$ and $t_1\underline{x}_2 \sim \underline{x}_1z_2$ imply $z_1z_2 \sim t_1t_2$, so $\circ$ is commutative.

4. $\succsim$ is monotone Let $x_1, x_1', y_1, z_1, z_1' \in X_1$, $x_2, x_2', y_2, z_2, \in X_2$ be such that $[x_1x_2] \circ [y_1y_2] = [z_1z_2]$ and $[x_1'x_2'] \circ [y_1y_2] = [z_1'z_2]$. That is, $z_1\underline{x}_2 \sim \underline{x}_1t_2$, $z_1'\underline{x}_2 \sim \underline{x}_1t_2'$, $t_1\underline{x}_2 \sim \underline{x}_1z_2$, $z_1't_2' \sim x_1'x_2'$, $z_1t_2 \sim x_1x_2$ and $t_1z_2 \sim y_1y_2$, for some $t_1 \in X_1$ and $t_2, t_2' \in X_2$. Suppose first that $x_1x_2 \succsim x_1'x_2'$. By the Thomsen condition, $z_1'\underline{x}_2 \sim \underline{x}_1t_2'$ and $z_1\underline{x}_2 \sim \underline{x}_1t_2$ imply $z_1't_2 \sim z_1t_2'$. By definition and hypothesis, $z_1t_2 \sim x_1x_2 \succsim x_1'x_2' \sim z_1't_2'$, coordinate independence consistency and coordinate independence imply $z_1z_2 \succsim z_1'z_2$. Conversely, if $z_1z_2 \succsim z_1'z_2$, then by coordinate independence and definition, $z_1\underline{x}_2 \succsim z_1'\underline{x}_2$ and $\underline{x}_1t_2 \succsim \underline{x}_1t_2'$. Using coordinate independence twice and transitivity, $z_1t_2 \succsim z_1't_2'$, that is $x_1x_2 \succsim x_1'x_2'$. By what we have shown, $[x_1x_2] \succsim [x_1'x_2']$ if and only if $[x_1x_2] \circ [y_1y_2] \succsim [x_1'x_2'] \circ [y_1y_2]$.

5. $(S, \circ, \succsim)$ **is bounded** Boundedness follows from the fact that $\succsim$ is essentially bounded. Moreover, it is clear from the definition of $\circ$ that $[\underline{x}_1\underline{x}_2]\circ[\bar{x}_1\bar{x}_2] = [\underline{x}_1\bar{x}_2] = [\bar{x}_1\underline{x}_2]$.

6. $\circ$ **is bisymmetric** In a commutative mean groupoid, the bisymmetric equation $(a\circ b)\circ(c\circ d) = (a\circ c)\circ(b\circ d)$ is equivalent to $(a\circ b)\circ(c\circ d) = (a\circ d)\circ(c\circ b)$. By definition of $\circ$, the latter is equivalent to $[x_1x_2] \circ [y_1y_2] = [x_1y_2] \circ [y_1x_2]$ for all $x_1, y_1 \in X_1$, and $x_2, y_2 \in X_2$. First, we will show that $[x_1x_2] \circ [\underline{x}_1\underline{x}_2] = [x_1\underline{x}_2]\circ[\underline{x}_1x_2]$. By definition, $[x_1x_2]\circ[\underline{x}_1\underline{x}_2] = [y_1\underline{x}_2]$ and $[x_1\underline{x}_2]\circ[\underline{x}_1x_2] = [z_1z_2]$, that is, $y_1\underline{x}_2 \sim \underline{x}_1y_2$, $z_1\underline{x}_2 \sim \underline{x}_1t_2$, $t_1\underline{x}_2 \sim \underline{x}_1z_2$, $z_1t_2 \sim x_1\underline{x}_2$, $t_1z_2 \sim \underline{x}_1x_2$, and $y_1y_2 \sim x_1x_2$, for some $t_1 \in X_1$ and $t_2, y_2 \in X_2$. We need to show that $z_1z_2 \sim y_1\underline{x}_2$.

It is clear from boundedness and coordinate independence that $\underline{x}_1\bar{x}_2 \sim \bar{x}_1\underline{x}_2 \succsim x_1\underline{x}_2$, $\underline{x}_1x_2 \succsim \underline{x}_1\underline{x}_2$. By monotonicity and idempotency,

$$[\underline{x}_1\bar{x}_2] = [\bar{x}_1\underline{x}_2] \succsim [z_1z_2] \succsim [\underline{x}_1\underline{x}_2],$$

by restricted solvability, we can find $s_1 \in X_1$ and $s_2 \in X_2$ such that $z_1z_2 \sim \underline{x}_1s_2 \sim s_1\underline{x}_2$. The Thomsen condition, respectively, on $z_1\underline{x}_2 \sim \underline{x}_1t_2$ and $t_1\underline{x}_2 \sim \underline{x}_1z_2$, $z_1z_2 \sim s_1\underline{x}_2$ and $z_1t_2 \sim x_1\underline{x}_2$, $z_1z_2 \sim \underline{x}_1s_2$ and $t_1z_2 \sim \underline{x}_1x_2$, yield respectively, $z_1z_2 \sim t_1t_2$, $x_1z_2 \sim s_1t_2$, $z_1x_2 \sim t_1s_2$.

Note that the Thomsen condition on $\underline{x}_1\bar{x}_2 \sim \bar{x}_1\underline{x}_2$ and $t_1\underline{x}_2 \sim \underline{x}_1z_2$ yield $t_1\bar{x}_2 \sim \bar{x}_1z_2$. By boundedness and coordinate independence, $t_1\bar{x}_2 \succsim t_1s_2 \succsim t_1\underline{x}_2$, that is $\bar{x}_1z_2 \succsim t_1s_2 \succsim \underline{x}_1z_2$. Thus, by restricted solvability, there exists $r_1 \in X_1$ such that $t_1s_2 \sim r_1z_2$. Similarly, there exists $r_2 \in X_2$ such that $s_1t_2 \sim z_1r_2$.

Now, by repeated application of the Thomsen condition on $z_1z_2 \sim t_1t_2$, $x_1z_2 \sim s_1t_2$, $z_1x_2 \sim t_1s_2$, $t_1s_2 \sim r_1z_2$ and $s_1t_2 \sim z_1r_2$, we have $x_1x_2 \sim r_1r_2 \sim s_1s_2$. Hence, $y_1y_2 \sim s_1s_2$. Since the Thomsen condition on $\underline{x}_1s_2 \sim s_1\underline{x}_2$ and $y_1\underline{x}_2 \sim \underline{x}_1y_2$ imply $s_1y_2 \sim s_1y_2$, coordinate independence consistency and coordinate independence yield $y_1\underline{x}_2 \sim s_1\underline{x}_2 \sim z_1z_2$, as desired.

It is not difficult to see that $[x_1x_2] \circ [\underline{x}_1\underline{x}_2] = [x_1\underline{x}_2] \circ [\underline{x}_1x_2]$ is equivalent to

$$[\underline{x}_1\underline{x}_2] \circ ([x_1x_2] \circ [y_1y_2]) = ([\underline{x}_1\underline{x}_2] \circ [x_1x_2]) \circ ([\underline{x}_1\underline{x}_2] \circ [y_1y_2]).$$

By boundedness and monotonicity, for all $x_1x_2 \in X_1 \times X_2$,

$$[\underline{x}_1\underline{x}_2] \circ [\bar{x}_1\bar{x}_2] \succsim [\underline{x}_1\underline{x}_2] \circ [x_1x_2] \succsim [\underline{x}_1\underline{x}_2] \circ [\underline{x}_1\underline{x}_2],$$

that is, by definition and idempotency,

$$[\underline{x}_1\bar{x}_2] = [\bar{x}_1\underline{x}_2] \succsim [\underline{x}_1\underline{x}_2] \circ [x_1x_2] \succsim [\underline{x}_1\underline{x}_2].$$

We can conclude that restricted resolvability applies to some representative of $[\underline{x}_1\underline{x}_2] \circ [x_1x_2]$. Thus, by a reasoning similar to the above, it can be shown that $[\underline{x}_1\underline{x}_2] \circ ([x_1x_2] \circ [y_1y_2]) = [\underline{x}_1\underline{x}_2] \circ ([x_1y_2] \circ [y_1x_2])$ and cancellation yields the desired result.

7. $(S, \circ, \succsim)$ **is Archimedean** $(S, \circ, \succsim)$ is Archimedean as $(X_1 \times X_2, \succsim)$ is Archimedean.

By Theorem 1 and the fact that $[x_1x_2] \circ [\underline{x}_1\underline{x}_2] = [x_1\underline{x}_2] \circ [\underline{x}_1x_2]$, there exists a nonempty set of functions $\mathcal{F}$ such that, for all alternatives x and y,

(i) $x \succsim y \Leftrightarrow f(x) \geq f(y), \ \forall f \in \mathcal{F}$;

(ii) $f(x) = f(x_1, \underline{x}_2) + f(\underline{x}_1, x_2), \ \forall f \in \mathcal{F}$;

(iii) $f(\underline{x}) = 0$ and $f(\bar{x}) = 1$, $\forall f \in \mathcal{F}$.

Uniqueness follows from Proposition 1. $\square$

In the following we suppose that X has three or more factors and in the following proofs we focus on two or three factors. It will be assumed that the other factors have arbitrary, but fixed, values which because of coordinate independence are immaterial, and so they will be suppressed from the notation. The induced relation $\succsim_{ijk}$ on factors $X_i \times X_j \times X_k$ with values fixed on the other factors is denoted $\succsim$, that on factors $X_i \times X_j$ by $\succsim_{ij}$, and that on X_i by $\succsim_i$. Note that, as $\succsim$ is a preorder, then the induced relations are also.

Lemma 6. *Suppose that $n \geq 3$ and Let $\succsim$ be an essentially bounded binary relation on $\prod_{i=1}^n X_i$ which satisfies restricted solvability and coordinate independence. Then, for $i, j = 1, \ldots, n$, $i \neq j$, the relation $\succsim_{ij}$ satisfies the Thomsen condition.*

Proof. Let $i, j = 1, \ldots, n$, be such that $i \neq j$ and suppose first that, $x_i\underline{x}_j \sim_{ij} \underline{x}_iy_j$ and $\underline{x}_ix_j \sim_{ij} y_i\underline{x}_j$. By definition and coordinate independence, $x_i\underline{x}_j\underline{x}_k \sim \underline{x}_iy_j\underline{x}_k$ and $\underline{x}_ix_j\underline{x}_k \sim y_i\underline{x}_j\underline{x}_k$. By boundedness and restricted solvability, there exists $x_k \in X_k$ such that $x_i\underline{x}_j\underline{x}_k \sim \underline{x}_i\underline{x}_jx_k \sim \underline{x}_iy_j\underline{x}_k$. By coordinate independence, $x_ix_j\underline{x}_k \sim \underline{x}_ix_jx_k \sim y_i\underline{x}_jx_k \sim y_iy_j\underline{x}_k$, hence $x_ix_j \sim_{ij} y_iy_j$. Second, we suppose that $x_iv_j \sim_{ij} v_iy_j$ and $v_ix_j \sim_{ij} y_iv_j$. By boundedness and restricted solvability, there exists $x_k \in X_k$ such that $v_i\underline{x}_j\underline{x}_k \sim \underline{x}_i\underline{x}_jx_k$; by what we have shown, as $\succsim$ is essentially bounded, $v_i\underline{x}_k \sim_{ik} \underline{x}_ix_k$ and $\underline{x}_i\bar{x}_k \sim_{ik} \bar{x}_i\underline{x}_k$ imply $v_i\bar{x}_k \sim_{ik} \bar{x}_ix_k$. $\succsim$ is essentially bounded, by coordinate independence $\bar{x}_ix_k \succsim_{ik} x_ix_k \succsim_{ik} \underline{x}_ix_k$. That is, $v_i\bar{x}_k \succsim_{ik} x_ix_k \succsim_{ik} v_i\underline{x}_k$; applying restricted solvability, there exists $y_k \in X_k$ such that $x_ix_k \sim_{ik} v_iy_k$. Then, by coordinate independence and hypothesis, $x_iv_jx_k \sim v_iv_jy_k \sim v_iy_jx_k$. Finally, by coordinate independence, $x_ix_jx_k \sim v_ix_jy_k \sim y_iv_jy_k \sim y_iy_jx_k$, hence $x_ix_j \sim_{ij} y_iy_j$. $\square$

Proof of Theorem 3. By Lemma 6, each $(X_i \times X_j, \succsim_{ij})$ satisfies the conditions of Theorem 2 and therefore, from what has been shown, each $(X_i \times X_j, \succsim_{ij})$ give rise to an Archimedean, bounded, partially ordered commutative mean groupoid $(X_i \times X_j / \sim_{ij}, \circ_{ij}, \succsim_{ij})$. It is readily seen from the definition of $(X_i \times X_j / \sim_{ij}, \circ_{ij}, \succsim_{ij})$ that, keeping a coordinate fixed, we can define a commutative mean groupoid on each coordinate X_i and X_j respectively. By coordinate independence and coordinate independence consistency, it is immaterial which X_j we choose to define the mean groupoid operation on X_i. Therefore, the mean groupoid defined pointwise on $X_i \times X_j \times X_k / \sim$ is an Archimedean, bounded, partially ordered commutative mean groupoid. A simple induction extends this result to any n. So, it can be deduced that there exists a nonempty set of functions $\mathcal{F}$ such that, for all alternatives x and y,

(i) $x \succsim y \Leftrightarrow f(x) \geq f(y), \ \forall f \in \mathcal{F}$;

(ii) $f(x) = \sum_{i=1}^{n} f(\underline{x}_1, \ldots, \underline{x}_{i-1}, x_i, \underline{x}_{i+1}, \ldots, \underline{x}_n), \ \forall f \in \mathcal{F}$;

(iii) $f(\underline{x}) = 0$ and $f(\bar{x}) = 1$, $\forall f \in \mathcal{F}$.

Uniqueness follows from Proposition 1. □

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